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A REVIEW OF THE SERIES IN THE SPECTRA OF THE ELEMENTS.

BY

F. A. SAUNDERS.

THE analysis of the spectra of the elements into series and systems of series, and the study of the relations among these within the spectrum of one element, and among different elements, is a subject which has received but little attention in spite of its obvious interest. For thirty years or more it has been known that the vibrations of the atoms are not haphazard, but are often arranged in a regular order, suggesting a series of overtones in acoustics. In certain cases, such as hydrogen, or zinc, this arrangement is quite conspicuous, and was recognized as soon as the full spectrum was obtained; in others, the spectra are so complex that there is little reason to hope that we may ever work out their structure.

The interest in the problem lies in the information which these studies may yield as to the nature of the atom and the physical processes going on within and about it, and in these days no information about the atom can be branded as useless. The difficulties which prevent its immediate solution are very numerous. The eye is sensitive to but a small part of the spectrum. For the investigations in question, the entire spectrum must be studied, from the shortest ultra-violet waves to the longest ultra-red; and several different experimental methods and much elaborate apparatus have to be employed in order to yield such a wide sweep of spectrum. Moreover, we do not possess means of agitating the atom which are sufficiently vigorous to make it

give its "complete" spectrum; in fact, such a word may have a very vague meaning. Some light-sources yield a little more than others, but, at the best, even with a variety of sources, we have to be content with a very imperfect view of the domain we wish to survey.

Certain comparatively recent discoveries will be of great assistance when applied to this problem. If the source of light is placed in a strong magnetic field, each line of the spectrum breaks up into a little group of lines (Zeeman effect), the arrangement of which appears to be characteristic of the series to which the line belongs. Something similar happens, at least in certain cases, with electric fields (Stark effect). But these effects cannot be observed for faint lines, and, unfortunately, the majority of the lines in all spectra seem to be faint.

The present state of the experimental study of spectrum series is one of flux, and whatever may be written about it now will probably be hopelessly out of date in half a dozen years. Something of the same sort may be said for the theoretical side of the question. Bohr, Sommerfeld, and others have built up a model of a simple atom whose unconventional actions would have shocked physicists of twenty years ago, but which seem to be capable of yielding the observed spectra. At the present writing, this very remarkable success has been obtained only for the simplest type of atoms, containing merely the nucleus, and one electron. This limits it to the spectrum of hydrogen and the spectrum of ionized helium. Other spectra, such as that of ordinary helium, are being attacked, but it is too early to say with what success.

The present article will not include any account of the theoretical investigations. These have been treated by Sommerfeld in a book, and are being made accessible to English readers in other ways. No comprehensive account of the experimental side of the problem exists. The beginnings of it may be found in Kayser's monumental "Handbook of Spectroscopy," but that book is now quite out of date on this branch of the subject. A book is to be published shortly by Professor A. Fowler, of London, which will give us an authoritative summary by one who has contributed largely to its development. In anticipation of this work, however, a brief attempt will be made in this article to outline our present knowledge of spectrum series. The matter

will be presented by chemical families, in the order of the simplicity of their spectra, which is nearly the same as the historical order of their solution. The various laws of series will be found under the elements in whose spectra they first occur.

Hydrogen.—There are two spectra of hydrogen, the simple, or series spectrum, and the compound spectrum, recently shown by G. P. Thomson to be due to the hydrogen molecule. The simple or atomic spectrum consists of one series, occurring in at least three different parts of the spectrum, as though transplanted unchanged. The frequency-differences are what remain unchanged in such cases; such series are called parallel series. Any series consists in itself of an arrangement of lines whose spacing diminishes regularly, approaching a limit whose frequency is finite, and is sometimes called the "convergence-frequency" of the series. The intensities of the lines fade away the nearer the limit is approached, and usually so rapidly that only very few of the lines can be observed; in laboratory sources it is rare that more than 15 lines of a series can be observed; the maximum number is about 50, and was obtained by R. W. Wood from the absorption of sodium vapor.

It is usual to speak, not of wave-lengths, but of their reciprocals, called wave-numbers, these being the number of waves per cm; and it will be convenient to define the "term" of a series line as the difference in wave-number between the limit of the series and the individual line. Thus, the hydrogen series yields a sequence of terms, which occurs unchanged in each repetition of the series, though the limits in the three cases are very different.

The visible portion of the spectrum consists of four lines of the "Balmer" series, the rest of which is in the ultra-violet. This series can be represented by the formula

$$\frac{1}{\lambda} = N\left\{\frac{1}{2^2} - \frac{1}{m^2}\right\}$$

where N is the so-called "universal series constant," having a value near 109678.2, and m takes in succession the integral values 3, 4, 5, etc. This formula yields the observed wave-numbers to quite an extraordinary degree of accuracy, though if the utmost refinement be demanded, there should be a correction term, of a nature indicated by theory. The lines are, moreover, actually very close doubles, whose frequency-differences are now known to remain approximately constant throughout the series.

In the extreme ultra-violet region, the series

$$\frac{1}{\lambda} = N\left\{\frac{1}{1^2} - \frac{1}{m^2}\right\}$$

occurs, discovered by Lyman, and beginning with the line λ1216. Paschen has found three lines in the ultra-red which belong to the series

$$\frac{1}{\lambda} = N\left\{\frac{1}{3^2} - \frac{1}{m^2}\right\}$$

Other series may exist in the deeper ultra-red, but these are very faint. The whole set of series, and therefore the whole spectrum of atomic hydrogen, can be represented by the formula

$$\frac{1}{\lambda} = N\left\{\frac{1}{n^2} - \frac{1}{m^2}\right\}$$

where, in general, the lines are stronger the smaller are the integers n and m. The smallest value for n is 1, and for m is always the integer next larger than n. The Bohr theory has had its greatest success in the explanation of this spectrum.

The Sodium Family.—The elements of this family yield spectra which are closely similar to one another, and are one step less simple than that of hydrogen. Each consists of four chief series, and each line in each series is double. The four series are known as principal, diffuse or first subordinate, sharp or second subordinate, and fundamental. The latter are called the "Bergmann" series by German writers, though improperly, as Fowler was the first to find one of this type, and Runge the first to point out their true relationships. These four type-series have certain characteristics which are worth noting. The *principal* series contain the chief line of the spectrum (the D line in the case of Na), which is produced in flames, and readily reversed in hotter sources, due to self-absorption. The separation of the pairs is not constant in this series, but they shrink rapidly in passing out toward the limit. The lines of a principal series pair have slightly unequal intensities, the stronger one being on the side of the shorter wave-length. The *diffuse* and *sharp* series have constant wave-number differences in each pair, and equal to that for the first pair of the principal series; their lines have an aspect well indicated by their names; and these two series converge to a common limit. The stronger line in each pair is here the one of *longer* wave-length. They require higher temperatures

or more vigorous excitation for their production than does the principal series. The *fundamental* series is a faint series of narrow pairs, lying in these cases in the ultra-red; its name is somewhat misleading.

Any of these series may be represented approximately by a series formula, of which there are several types. The general type is

$$\frac{1}{\lambda} = \frac{N}{[n + a + f(n)]^2} - \frac{N}{[m + b + f(m)]^2}$$

where N is the universal series constant, as before, though there are indications that its value should change slightly; n and m are variable integers, n having a single value for the whole of a series, while m takes a set of integral values in succession; a and b are constants, and the functions $f(n)$ and $f(m)$ may take many forms, none of them quite satisfactory. Fair success is obtained by giving the functions the form c/n^2 and d/m^2, c and d being new constants, but in many cases no simple form can possibly serve for these functions. The portion of the denominator written as $a + f(n)$ is sometimes referred to for brevity as the "residual"; it should settle down to a constant value as we proceed toward the limit of the series, and usually does so with rapidity. These formulæ have as yet no very sound theoretical basis, nor any established and universal shape; hence it has proved convenient to dodge the question of how they should be written by using abbreviations. In the above general formula the right-hand half of the expression for the wave-number is what we have already called the term, and we indicate it by the symbol (m,b), no matter how complex the formula actually used might be. The letter bracketed with m in this symbol is usually chosen to indicate the series to which it refers; thus, the letters p,s,d and f refer respectively to the principal, sharp, diffuse and fundamental series; for example, (1,p) stands for the first term of the principal series, (3,s) for the third term of the sharp. In the case of such elements as have series of other sorts beside pairs, the letters indicate the sort; thus in such elements, for example in Ca, small letters are used for series of triplets, large letters for single-line ("singlet") series, and Greek letters for pair series, except that German authors have so far used old Gothic letters instead of Greek. These distinctions are, naturally, not to be made, and have

no meaning in the case of the elements of the sodium family, where all series are pairs.

The variable integer m has its lowest value for the first line of the series. In no cases but those of hydrogen and ionized helium can this integer be definitely settled as yet; in fact, since m is always associated with an added constant whose value is anything we please, the value of m becomes equally adjustable. German writers use for the first line of the series the value $m = 2$ in the case of the principal series, 3 for the diffuse and 4 for the fundamental, while for the sharp they use values intermediate between integers, namely 1.5, 2.5, 3.5, etc. Some English-speaking writers prefer to put m = 1 for all first lines; such theoretical indications as there are rather favor the German custom.

A principal series may be written, in the symbolic way explained above, $1/\lambda = (1,s) - (m,p)$. The limit of the series is the term (1,s), which is the first term of the sharp series, as the letter indicates. This connection between the principal and sharp series amounts to saying that they have a first member in common. On this idea, the first line of the principal series has the wave-number (1,s)−(1,p), while the first line of the sharp series is (1,p)−(1,s). The latter is negative, and it might well be objected that a negative wave-number is meaningless. The Bohr theory offers a possible explanation of this curious case, and others of the same sort, as indicated below. The *Rydberg-Schuster law* says that the wave-number-difference between the limit of the principal series and the limit of the sharp and the diffuse series (which they share in common) is equal to the wave-number of the first line of the principal series. If this is otherwise expressed it amounts to saying that the limit of the sharp series is the first term of the principal. This fact was used above in writing the first line of the sharp series as (1,p)−(1,s). The whole of this series would be indicated by $1/\lambda = (1,p) - (m,s)$.

The diffuse series is given by the formula $1/\lambda = (1,p) - (m.d)$. The terms (m,d) and (m,p) are each double in the case of pair series, and in reality the diffuse pair contains three lines, the first and stronger line having a faint satellite. This is noticeable only in the case of wide pairs (*e.g.*, Rb and Cs), and even then only in the first few series members, for the reason that the satellite and the major line close up together.

The fundamental series is given by $1/\lambda = (1,d) - (m,f)$. Its

limit is the first term of the diffuse series, a law due to Runge. The terms (m,f) are also double, though very close in most cases. Since the two values of (1,d) are not far apart, this series consists of narrow pairs, of which each line is itself a close double. While this complexity has been found only for Cs, there is every reason to believe that it is general. We are dealing here with what is generally called the "fine structure" of the lines, and a great many series lines prove to be complex when examined under the highest resolving-powers. In most cases this complexity is so difficult to observe that we may safely ignore it.

Ritz formulated a so-called "combination law" to the effect that any two terms, chosen from among the lists of those in the four type-series, might be combined, and their difference would give a wave-number which might correspond to a real line. Combinations which occur more or less commonly are as follows: (1,p)−(m,p); (2,s)−(m,p); (2,p)−(m,d); (2,p)−(m,s); (2,d)−(m,f); etc. Of these the first, but only the first, are similar to difference tones in acoustics. In such elements as have systems of triplet and singlet series (*e.g.*, Zn, Ca, etc.), there also occur intersystem combinations, of which the most important is (1,S)−(m,p_2). Here the (m,p) terms are triple, but the combination occurs only with the middle value, indicated by the subscript.

On the basis of the modern theory, the Ritz combination principle can be given a simple physical interpretation. If the outer electrons in the atom can exist in a number of different stationary states, in each of which their energy has a definite value, then the radiation is supposed to occur on the transfer of an electron from one of these states to another. The change of energy which is involved in this event is proportional to the frequency of the light, and the light is supposed to be emitted during this process and at no other time. Thus, the "terms" should be proportional to the energy in the corresponding states, and their differences, which by the combination principle give the frequency of observed spectrum lines, are proportional to the changes in energy associated with the transfer of an electron from one state to another. On this theory, if a negative wave-number occurs, it indicates merely that the electron passes from one state to the other in a direction opposite to that which we might expect on spectroscopic grounds, and this idea involves no special difficulty.

The entire spectrum of Li,Na,K,Rb or Cs could then be written as $1/\lambda=(1,s)-(m,p)$; $1/\lambda=(1,p)-(m,s)$; $1/\lambda=(1,p)-(m,d)$;

$1/\lambda = (1,d) - (m,f)$, together with combinations formed from these terms. Most of the combinations which one might test are in the ultra-red and are not observed. All combination lines are likely to be faint. Exactly the same combinations have not been observed in all these spectra, but it seems hardly worth while to give the details in the separate cases.

In comparing the spectra of the elements in this group it is interesting to note that the widths of the pairs is proportional approximately, though not exactly, to the square root of the atomic number of the element. It might also be remarked that the spectrum as a whole shifts toward the slower vibrations with increasing atomic weight, as one might expect from mechanical considerations. These laws appear to be general.

All these elements probably possess another spectrum as yet practically unknown. Traces of it have been found in the heavier elements. This has been called the spark, or enhanced spectrum, but it might better be called the spectrum of the ionized element. It is particularly easy with the elements of this family to detach one electron from the atom. The new spectrum would appear when *two* electrons were detached temporarily, and one of these returned, thus giving us the optical vibrations of the atom minus one electron. As the atom in this condition would probably have a structure resembling the inert element to the left in Mendelejeff's table, *i.e.*, to Neon, Argon, etc., it is not surprising that it is difficult to tear off a second electron, and obtain the spectrum in question. It is probable that the "Grundspektra" of Goldstein are mixtures of this spectrum with the ordinary one.

The Calcium-Zinc Family.—Passing to the next column of Mendelejeff's table of the elements, we find more complicated spectra, which are not as yet quite completely understood. Each contains at least four type series in each of two systems, in one of which the lines are all triplets, in the other singlets. In addition, the spectrum of the ionized element is rather readily produced, giving rise to an independent but similar system, this time of pairs, as one might expect, since, with one electron gone, the atomic structure of this family should resemble that of the sodium family. In each system combination series occur, and they are apparently more important with the heavier elements, which give the richer spectra. Inter-system combination series are here first met with, as mentioned above. There are other features of these spectra as yet unexplained, and in some cases our knowledge is

far from complete. For instance, well known as the spectrum of Hg is supposed to be, its pair system has not yet been studied.

The Inert Gases, He, Ne, etc.—The spectrum of *Helium* likewise contains two related systems of series, in this case observed as pairs and singlets, but perhaps really more complex, as well as an unrelated system (pairs) of lines produced by ionized helium. The atom of He probably possesses two external electrons, or perhaps only two altogether. On this account its spectrum ought to have a form similar to that of the other elements which have this structural feature, *i.e.*, to Mg, etc. This is the case. If one of these electrons were removed, the atomic structure should bear an external resemblance to that of hydrogen, and in fact the spectrum of ionized He consists of one series, following the Balmer formula, and repeated, exactly as is that of H, in two other parts of the spectrum, with fair prospects of the discovery, eventually, of still more of these parallel series. There is a difference, however, shared by the series of ionized Mg, Ca, etc., that 4N must be used instead of N itself in the series formula, and Bohr's theory explains this as due to the presence of the extra, unbalanced charge on the nucleus of the atom.

When we pass to Neon, however, we are dealing, following the fruitful atomic theory of Lewis and Langmuir, with a cubical atom, whose vibrations would be expected to be very complex. As a matter of fact this is the case, and only recently has this spectrum been successfully attacked, in a masterly memoir by Paschen. He finds one series system, but each "term" of each series is really very complex, consisting perhaps of ten widely separated lines, shrinking, however, on passing out along the series, though not in any simple manner. The spectrum of ionized Neon has not yet been identified, and the complexity of its ordinary spectrum is such that it is doubtful if we yet know all about it. Argon is equally rich in lines, and the study of its spectrum is apparently yielding results somewhat similar to those for Ne.

The Cu, Ag, Au Group.—These spectra combine the features of the Na group (pair system) with some of the characteristics of Ne and A, and they have not yet been adequately studied. The spectra of the ionized elements have not been clearly isolated.

The Al, In, Tl Group.—Passing next to the Al group, we return to pair systems as in Na, but the spectra of the ionized elements contain systems of triplets and singlets, which one would expect by analogy to resemble the ordinary (un-ionized)

spectra of the Ca group. The pair systems are fairly well known, the others practically unknown; they lie among the short wavelengths, as one might anticipate from the fact that the forces in an unbalanced system should be greater, and therefore the rates of vibration more rapid.

The Elements of the Carbon Group.—The spectra of C and Si show definite signs of series, but none have as yet been published. Analogy indicates that the ordinary spectra should contain two related systems of series, perhaps of singlets and triplets, and that the ionized element should give a single system, perhaps of pairs. The latter spectrum should be easy to obtain, and therefore difficult to separate from the former.

The other elements in this column show parallel groups of lines, scattered at what seem to be random intervals. The analysis of the spectrum of Neon, which seemed to contain such groups, but was shown to be made up of series, leads us to hope that a similar result will be found to hold in these cases also, but the work has not yet been carried out. These groups constitute what has been called the " second type of regularity " in spectra.

The Elements of the Nitrogen Group.—The same remarks may be made in regard to this family of elements as apply to the preceding, except that one would expect the arrangement of the ordinary spectrum and the spectrum of the ionized element to be interchanged. No series have as yet been worked out here, but obvious triplets and pairs occur in N. The heavier elements contain parallel groups.

The Elements of the Oxygen Group.—The Oxygen family contains two systems of series, pairs and triplets, which seem both to come from the ordinary atom. Apparently the spectrum of the ionized element is the " spark " spectrum, which is well known, but rather complex, and in which no regularities have as yet been found.

The Elements of the Chlorine Group.—These give complex spectra most of which have not yet shown any systematic arrangement, though Paschen has very recently shown that ionized Cl gives series of triplets.

The Fe-Pt Elements.—These elements also give very complex spectra, though a patient analysis of a few of them has yielded parallel groups. It seems at present quite hopeless to attempt to arrange these spectra into series.

This brief survey of our knowledge of spectrum series serves

to show quite clearly how limited that knowledge is, and how urgent is the need for more workers in this important field. Every new source of light, or mode of excitation, at once opens up an extensive field of experimental investigation, as it should be tried out for all possible elements in all possible parts of the spectrum. Last year at least two such modes of excitation were developed which have great interest: the high-vacuum spark used by Millikan to break new ground among the shortest wavelengths, and the method of explosion of very thin wires used by Anderson. Each of these promises to yield interesting new spectra, and should be most carefully studied from a series standpoint. Other methods will doubtless be developed in the future.

A general review of the series spectra of the elements yields two new laws, recently stated by Kossel and Sommerfeld on the basis of evidence still incomplete, though their probable validity has long been recognized.

The first of these is that the spectrum of an ionized element (lacking one electron) is like that of the element in the next column to the left in Mendelejeff's table, though shifted toward higher frequencies; for example, ionized Ca (Ca^+) gives a spectrum very like that of K; and this would be expected since each has but one external electron.

The second law states that the spectra alternate in character in passing across Mendelejeff's table. Thus the spectra of Na and Al consist of pair series and resemble each other closely; N is probably similar. Ca, Si and S have many features in common, and each probably furnishes two sets of series, singlets and triplets. This could not be foreseen from considerations of atomic structure, but indicates that when electrons are added to the outside of an atom, as occurs in passing from left to right across the table, the electrons have a strong tendency to unite into rather inactive pairs; thus the spectrum of Al appears to be that due to a single external electron, though there must be three present; but two of these are so closely bound together that they do not take part in the emission of light. Langmuir has recently called attention to this tendency of electrons to form pairs, on the basis of quite different evidence. That other laws of general interest will come out of such study cannot be doubted, and it is earnestly hoped that the near future will witness a vigorous development of this subject.

JEFFERSON PHYSICAL LABORATORY,
HARVARD UNIVERSITY.

NOTE ON A FORMULA ASSOCIATED WITH THE PATH OF A RAY THROUGH A PRISM.

BY

H. S. UHLER.

PROFESSOR SOUTHALL has recently published a trigonometrical formula for the calculation of the angle of incidence of a ray which traverses a prism in a principal section and emerges under a prescribed angle of deviation.[1] With regard to this formula he makes the following comment: "It is possible that it may be arranged in a form more convenient for logarithmic computation." More than seven years ago the present writer published the solution of this problem for the case of a ray traversing a prism obliquely.[2] Accordingly it may be of practical interest to present, in this place, the logarithmic formulæ as simplified for a principal plane.

The angle α_0 is first calculated from the equation

$$\sin \alpha_0 = n \sin \tfrac{1}{2}\beta,$$

where α_0 and β denote respectively the angle of incidence corresponding to minimum deviation, and the refracting angle of the prism.

An auxiliary angle ϕ is next computed from the equation

$$\tan \phi = \tan c \cdot \sqrt{\left\{\sin\left[\tfrac{1}{2}(\epsilon + \beta) + \alpha_0\right] \sin\left[\tfrac{1}{2}(\epsilon + \beta) - \alpha_0\right]\right\}},$$

where c and ϵ symbolize respectively the critical angle, and the prescribed angle of deviation. If more convenient, $\tan c$ may be replaced by

$$1 \Big/ \sqrt{(n + 1)(n - 1)}.$$

The two values of the angle of refraction, α', may now be obtained by the aid of the following formula:

$$\tan (\alpha' - \tfrac{1}{2}\beta) = \pm \sin \phi \cot \tfrac{1}{2}(\epsilon + \beta).$$

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[1] James P. C. Southall: "Note on the path of a ray through a prism in a principal section." *Jour. Opt. Soc.*, **4**, p. 283, 1920.

[2] H. S. Uhler: "On the deviation produced by prisms." *Am. Jour. Sci.*, **35**, p. 396, 1913.

Finally

$$\sin \alpha = n \sin \alpha'.$$

In the illustrative case, taken by Professor Southall, where $n = 1.5$, $\beta = 60°$, and $\epsilon = 40°$ we find

$$\begin{aligned}
\alpha_0 &= 48° \; 35' \; 25.4'' \\
\phi &= 7° \; 56' \; 28.7'' \\
\alpha' - \tfrac{1}{2}\beta &= \pm (6° \; 36' \; 45.8'') \\
(\alpha')_+ &= 36° \; 36' \; 45.8'' \\
(\alpha')_- &= 23° \; 23' \; 14.2'' \\
(\alpha)_+ &= 63° \; 27' \; 27.7'' \\
(\alpha)_- &= 36° \; 32' \; 32.3''
\end{aligned}$$

The numerical values of α just given confirm those of Professor Southall absolutely.

SLOANE PHYSICS LABORATORY,
YALE UNIVERSITY,
Nov. 29, 1920.

NOTE ON MR. T. SMITH'S METHOD OF TRACING RAYS THROUGH AN OPTICAL SYSTEM.

BY

JAMES P. C. SOUTHALL.

IN the *Proceedings of the Physical Society of London* and partly also in the *Transactions of the Optical Society* Mr. T. Smith of the National Physical Laboratory at Teddington, England, has recently contributed a series of valuable papers "On tracing rays through an optical system" which I am sure will be of special interest to such readers of this JOURNAL who, like myself, are particularly concerned with the theory and design of optical instruments. In all the various schemes for the trigonometrical computation of the path of a ray refracted at a spherical surface the ray itself is determined by two arbitrary parameters, sometimes called the "ray-coördinates"; and if one method possesses any decided advantage over another, it will nearly always be found to be due chiefly to a more convenient choice of these controlling factors. One of these parameters must be a linear magnitude, for example, the intercept of the ray on the optical axis or on some other prescribed line; whereas the other parameter may be, and indeed usually is, an angular magnitude; the angle which is most often used for this purpose being the slope of the ray or its inclination to the axis (which is denoted in the following by the symbol θ). In the scheme which Mr. Smith recommends the ray-parameters are this angle θ and the length of the perpendicular let fall from the vertex A of the spherical refracting surface on to the ray in question. One advantage of using this perpendicular is that the formulæ obtained apply equally well to a plane refracting surface; and it may be remarked in passing that other writers have made use of this parameter also, among whom may be mentioned A. Kerber, who has proposed several systems of formulæ for tracing rays through an optical instrument. If the feet of the perpendiculars let fall from A on to the incident ray and its corresponding refracted ray are designated by T, T′, and if the lengths of these perpendiculars are denoted by g, g', then $g = AT$, $g' = AT'$. Let the slopes of

these rays be denoted by θ, θ' and the angles of incidence and refraction by α, α'; and moreover let R denote the curvature of the spherical refracting surface; and n, n′ the indices of refraction of the two media; then Mr. Smith's formulæ, as published by him originally in the *Proceedings of the Physical Society*, xxx (1918), 221–223, written, however, in terms of the above symbols, may be given as follows:

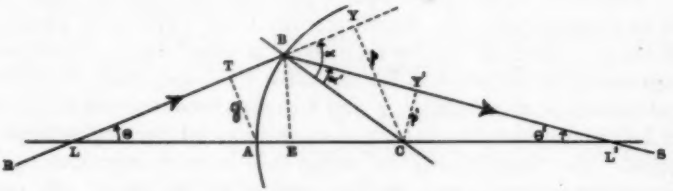
$$\sin \alpha = \sin \theta + g \cdot R,$$

$$\sin \alpha' = \frac{n}{n'} \sin \alpha, \quad \theta' = \theta + \alpha' - \alpha,$$

$$g' = \frac{\cos \alpha' + \cos \theta'}{\cos \alpha + \cos \theta} g.$$

Now in practically all these schemes of calculation the process involves, as here, the application of the law of refraction as expressed by the equation $n'.\sin\alpha' = n.\sin\alpha$ at each surface of

FIG. 1.



the lens-system in succession, whereby the angle of refraction (α') may be ascertained, and consequently the slope (θ') of the refracted ray can be found by use of the invariant-relation $\theta' - \alpha' = \theta - \alpha$. And in fact it is right at this stage of affairs that much of the labor of such calculations occurs, inasmuch as tables have to be employed and interpolations made in order to obtain the values (in degrees, minutes and seconds or preferably in radians) of the angles denoted by α, α' and θ. In his last paper on this subject (*Proc. Phys. Soc.*, xxxii (1920), 252–264) Mr. Smith seeks to find some substitution for the third step ($\theta' = \theta + \alpha' - \alpha$) in order to avoid the tedious procedure of evaluating these angles and in the hope of thereby effecting a material

shortening of the entire process, as is devoutly to be wished by all who have much of this work to do. Whether he has succeeded in this latter purpose appears to me doubtful; in fact, I myself have long been of the opinion that it was almost a waste of effort to try to devise essentially new systems of ray-calculation, that any really satisfactory method is on the whole about as economical as another, and that in all of them there is a certain limiting number of actual and necessary arithmetical operations which cannot be made less by any algebraic artifice. Of course, some individuals will work faster than others, and particular types of calculating machines will be better adapted than others. I am even old-fashioned enough to raise the question whether a calculating machine is more expeditious in the long run than the in-and-out process of logarithms; and my own experience, after trying out many schemes of ray-tracing, is that just when one fondly imagines he has found a promising short cut here or there, almost invariably he encounters a compensating amount of difficulty or delay at some other quite unexpected stage of the proceedings.

However, about these things opinions may differ; and so let us examine what Mr. Smith actually does. The main feature of his new method consists essentially in obtaining a convenient trigonometrical expression for the ratio $g' : g$, such that, the sines and cosines of the angles α, α' and θ having been ascertained, it is no longer necessary to know the magnitudes of the angles themselves. Mr. Smith's way of doing this is most ingenious and interesting in every way, as the readers of his paper will see for themselves; but the same result can be obtained also by a comparatively simple trigonometrical transformation as follows:

$$\frac{g'}{g} = \frac{\sin\alpha' - \sin\theta'}{\sin\alpha - \sin\theta} = \frac{\sin\frac{\alpha'-\theta'}{2}\cdot\cos\frac{\alpha'+\theta'}{2}}{\sin\frac{\alpha-\theta}{2}\cdot\cos\frac{\alpha+\theta}{2}} = \frac{\cos\frac{2\alpha'+\theta-\alpha}{2}}{\cos\frac{\alpha+\theta}{2}},$$

since $\alpha' - \theta' = \alpha - \theta$. Now if numerator and denominator of the last fraction are multiplied, first, by $\cos\frac{1}{2}(2\alpha' - \theta + \alpha)$, and, second, by $\cos\frac{1}{2}(\alpha + \theta)$, we shall obtain thus two expressions for the ratio $g' : g$, which may evidently be written as follows:

$$\frac{g'}{g} = \frac{\cos 2\alpha' + \cos(\theta - \alpha)}{\cos(\alpha + \alpha') + \cos(\theta - \alpha')} = \frac{\cos(\alpha - \alpha') + \cos(\alpha' + \theta)}{1 + \cos(\alpha + \theta)};$$

and, hence, also:

$$\frac{g'}{g} = \frac{\cos 2\alpha' + \cos(\theta - \alpha) + \cos(\alpha - \alpha') + \cos(\alpha' + \theta)}{1 + \cos(\alpha + \theta) + \cos(\alpha + \alpha') + \cos(\theta - \alpha')}$$

$$= \frac{1 + \cos(\alpha + \alpha') + \cos(\theta - \alpha') + \cos(\theta + \alpha) + 2(\sin\alpha - \sin\alpha')(\sin\alpha' + \sin\theta)}{1 + \cos(\alpha + \alpha') + \cos(\theta - \alpha') + \cos(\alpha + \theta)}$$

$$= 1 + \frac{2(\sin\alpha - \sin\alpha')(\sin\alpha' + \sin\theta)}{1 + \cos(\alpha + \alpha') + \cos(\theta - \alpha') + \cos(\alpha + \theta)}.$$

It is easy to pass now to Mr. Smith's final form of this expression, as follows:

$$g' = g + \frac{4g(\sin\alpha - \sin\alpha')(\sin\alpha' + \sin\theta)}{(\sin\theta - \sin\alpha + \sin\alpha')^2 + (\cos\theta + \cos\alpha + \cos\alpha')^2 - 1}$$

wherein the only trigonometrical functions that occur are the sines and cosines of the angles α, α' and θ.

In a very interesting discussion of Mr. Smith's paper in the same number of the *Proceedings of the Physical Society* Mr. F. J. W. Whipple has communicated an elegant geometrical method of obtaining this result.

Thus, the system of formulæ for tracing the ray at the kth surface, given in the order of the operations, is as follows:

$$\sin\alpha_k = \sin\theta_k + g_k \cdot R_k,$$

$$n_{k+1} \cdot \sin\alpha'_k = n_k \cdot \sin\alpha_k,$$

$$g'_k = g_k + \frac{4g_k(\sin\alpha_k - \sin\alpha'_k)(\sin\alpha'_k + \sin\theta_k)}{(\sin\theta_k - \sin\alpha_k + \sin\alpha'_k)^2 + (\cos\theta_k + \cos\alpha_k + \cos\alpha'_k)^2 - 1}$$

$$\sin\theta_{k+1} = \sin\alpha'_k - g'_k \cdot R_k,$$

$$g_{k+1} = g'_k + d_k \cdot \sin\theta_{k+1} \quad (\text{where } d_k = A_kA_{k+1}).$$

(For a paraxial ray we have merely to write the angles themselves in place of the sines of the angles and put each of the cosines equal to unity.)

In order to avoid as much as possible all unnecessary reference to trigonometrical tables, all that is needed here is a simple table giving the cosine of the angle corresponding to a given value of the sine, and for this purpose a four-place table which can be printed all on one single sheet is quite sufficient. The accompany-

ing table is reproduced from Mr. Smith's paper. The interpolations, as he points out, can be made mentally without difficulty.

COSINES IN TERMS OF SINES

(Copied from *Proc. Phys. Soc.*, xxxii, opposite p. 258)

	0	1	2	3	4	5	6	7	8	9	D	D^{-1}
.70	.7141	.7132	.7122	.7112	.7102	.7092	.7082	.7072	.7062	.7052	−10	−10
1	.7042	.7032	.7022	.7012	.7001	.6991	.6981	.6971	.6960	.6950	−10	−10
2	.6940	.6929	.6919	.6908	.6898	.6887	.6877	.6866	.6856	.6845	−11	− 9
3	.6834	.6824	.6813	.6802	.6791	.6781	.6770	.6759	.6748	.6737	−11	− 9
4	.6726	.6715	.6704	.6693	.6682	.6671	.6659	.6648	.6637	.6626	−11	− 9
5	.6614	.6603	.6592	.6580	.6569	.6557	.6546	.6534	.6523	.6511	−11	− 9
6	.6499	.6488	.6476	.6464	.6452	.6440	.6428	.6416	.6404	.6392	−12	− 9
7	.6380	.6368	.6356	.6344	.6332	.6320	.6307	.6295	.6283	.6270	−12	− 8
8	.6258	.6245	.6233	.6220	.6208	.6195	.6182	.6170	.6157	.6144	−13	− 8
9	.6131	.6118	.6105	.6092	.6079	.6066	.6053	.6040	.6027	.6013	−13	− 8
.80	.6000	.5987	.5973	.5960	.5946	.5933	.5919	.5906	.5892	.5878	−14	− 7
1	.5864	.5851	.5837	.5823	.5809	.5795	.5781	.5766	.5752	.5738	−14	− 7
2	.5724	.5709	.5695	.5680	.5666	.5651	.5637	.5622	.5607	.5592	−15	− 7
3	.5578	.5563	.5548	.5533	.5518	.5502	.5487	.5472	.5457	.5441	−15	− 7
4	.5426	.5410	.5395	.5379	.5363	.5348	.5332	.5316	.5300	.5284	−16	− 6
5	.5268	.5252	.5235	.5219	.5203	.5186	.5170	.5153	.5136	.5120	−16	− 6
6	.5103	.5086	.5069	.5052	.5035	.5018	.5000	.4983	.4966	.4948	−17	− 6
7	.4931	.4913	.4895	.4877	.4859	.4841	.4823	.4805	.4787	.4768	−18	− 6
8	.4750	.4731	.4712	.4694	.4675	.4656	.4637	.4618	.4598	.4579	−19	− 5
9	.4560	.4540	.4520	.4501	.4481	.4461	.4441	.4420	.4400	.4379	−20	− 5
.90	.4359	.4338	.4317	.4296	.4275	.4254	.4233	.4211	.4190	.4168	−21	− 5
1	.4146	.4124	.4102	.4080	.4057	.4035	.4012	.3989	.3966	.3943	−23	− 4
2	.3919	.3896	.3872	.3848	.3824	.3800	.3775	.3751	.3726	.3701	−24	− 4
3	.3676	.3650	.3625	.3599	.3573	.3546	.3520	.3493	.3466	.3439	−26	− 4
4	.3412	.3384	.3356	.3328	.3299	.3271	.3242	.3212	.3183	.3153	−29	− 3
5	.3122	.3092	.3061	.3030	.2998	.2966	.2934	.2901	.2868	.2834	−32	− 3
6	.2800	.2765	.2730	.2695	.2659	.2622	.2585	.2548	.2510	.2471	−37	− 3
7	.2431	.2391	.2350	.2308	.2265	.2222	.2178	.2132	.2086	.2039	−44	− 2
8	.1990	.1940	.1889	.1836	.1782	.1726	.1667	.1607	.1545	.1479	−53	− 2
9	.1411	.1339	.1262	.1181	.1094	.0999	.0894	.0774	.0632	.0447	....	....
.990	.1411	.1404	.1397	.1389	.1382	.1375	.1368	.1361	.1353	.1346	....	....
1	.1339	.1331	.1324	.1316	.1309	.1301	.1293	.1286	.1278	.1270	....	....
2	.1262	.1254	.1247	.1239	.1231	.1222	.1214	.1206	.1198	.1190	....	....
3	.1181	.1173	.1164	.1156	.1147	.1138	.1130	.1121	.1112	.1103	....	....
4	.1094	.1085	.1075	.1066	.1057	.1047	.1038	.1028	.1018	.1009	....	....
5	.0999	.0989	.0979	.0968	.0958	.0948	.0937	.0926	.0916	.0905	....	....
6	.0894	.0882	.0871	.0859	.0848	.0836	.0824	.0812	.0799	.0787	....	....
7	.0774	.0761	.0748	.0734	.0721	.0707	.0692	.0678	.0663	.0648	....	....
8	.0632	.0616	.0600	.0583	.0565	.0548	.0529	.0510	.0490	.0469	....	....
9	.0447	.0424	.0400	.0374	.0346	.0316	.0283	.0245	.0200	.0141	....	....

As an actual numerical illustration will doubtless be of interest, I give below the results of the calculation of the path of a ray through a double cemented achromatic microscope objective composed of a symmetric convex (or equibiconvex) crown glass lens and a planoconcave flint glass lens, with numerical data as follows:

Indices: $n_1 = n_4 = 1$ (air); $n_2 = 1.5166$ (crown); $n_3 = 1.6256$ (flint).

Radii and curvatures: $r_1 = -r_2 = +0.35$ inch; $r_3 = \infty$.

$R_1 = -R_3 = +2.85714$; $R_3 = 0$.

Thicknesses: $d_1 = d_2 = +0.04$ inch.

The ray which is here calculated emanates from a point L_1 six inches from the vertex of the first surface ($A_1L_1 = v_1 = -6$),

and meets the first surface at a point which is 0.1 inch above the axis ($h_1 = +0.1$). The initial values of the ray-parameters are found from the formulæ:

$$\tan\theta_1 = -\frac{h_1}{v_1 - r_1 + \sqrt{r_1^2 - h_1^2}}, \quad g_1 = -v_1 \cdot \sin\theta_1.$$

The table below shows the values of the sines and cosines of the angles and the magnitudes of the linear parameters g, g' as found by Mr. Smith's system of formulæ:

	k=1	k=2	k=3
$\sin\theta_k$	+0.016624	−0.08943	−0.06313
$\sin\alpha_k$	+0.30160	−0.36749	−0.06313
$\sin\alpha'_k$	+0.19887	−0.34285	−0.10262
$\cos\theta_k$	+0.9999	+0.9960	+0.9980
$\cos\alpha_k$	+0.9535	+0.9300	+0.9980
$\cos\alpha'_k$	+0.9800	+0.9394	+0.9947
g_k	+0.099744	+0.097327	+0.095377
g'_k	+0.100904	+0.097902	+0.095063

If the emergent ray crosses the axis at L_4, and if we put $v'_3 = A_3L_4$, then $v'_3 = -g'_3/\sin\theta_4$, so that we find here:

$$v'_3 = +0.92636 \text{ inch.}$$

These calculations were made on a machine, and I found that it requires a rather large number of operations to obtain the values of g' for each surface.

It might be well to compare Mr. Smith's formulæ with the following system in which the magnitudes denoted by p and p' are the central perpendiculars on the incident and refracted rays:

$$p' = \frac{n}{n'}p, \quad \sin\alpha = pR, \quad \sin\alpha' = p'R',$$
$$\theta' = \theta + \alpha' - \alpha;$$

and in passing from the kth to the $(k+1)$th surface:

$$p_{k+1} = p'_k + a_k \cdot \sin\theta_{k+1},$$

where

$$a_k = d_k + r_{k+1} - r_k.$$

For the plane surface we shall need in this case a special set of formulæ, as follows:

$$v_3 = v'_2 - d_2, \quad \sin\theta_4 = n_3 \sin\theta_3 / n_4, \quad v'_3 = v_3 \cdot \tan\theta_3 / \tan\theta_4.$$

The initial value of the central perpendicular is given by $p_1 = (r_1 - v_1)\sin\theta_1$. In the above formulæ $v_k = A_kL_k$, $v'_k = A_kL_{k+1}$,

where L_k and L_{k+1} designate the points where the ray crosses the axis before and after refraction at the kth surface.

Before leaving this subject there is at least one other matter in Mr. Smith's paper which seems to me of more or less theoretical interest, and to which I think special attention should be called. In the accompanying diagram the centre of the spherical refracting surface is at the point marked C and the vertex at A, so that the optical axis is represented by the straight line AC and the step from A to C is equal to the radius ($r = AC$). A ray RB, incident on the surface at B, crosses the axis at the point marked L, and the corresponding refracted ray BS crosses the axis at L′; and in order that the following statements may be quite general and applicable to all cases, it should be noted that R is used here to designate any point in the first medium (of index n) which lies on the incident ray; and, similarly, S designates any point in the second medium (of index n′) through which the corresponding refracted ray passes. The abscissæ of L, L′ with respect to the centre C may be denoted by c, c', that is, $c = CL$, $c' = CL'$; and the "ray-lengths" measured from the point of incidence may be denoted by l, l' ($l = BL$, $l' = BL'$). These magnitudes are connected by the well-known invariant relation:

$$\frac{n'c'}{l'} = \frac{nc}{l}.$$

Draw the bisector of the angle RBS, and let E designate the point where it meets the optical axis, and put $EL = x$, $EL' = x'$; then, evidently, since BE is the bisector of the interior or exterior angle at B in the triangle LBL′, we must have:

$$x' : x = l' : l,$$

and hence also:

$$\frac{n'c'}{x'} = \frac{nc}{x}.$$

Now $c = x - e$, $c' = x' - e$, where $e = EC$, and if we substitute these expressions for c and c' in the last equation, we may write:

$$\frac{n'}{x'} = \frac{n}{x} + \frac{n' - n}{e}.$$

The similarity between this formula and the familiar abscissa-equation for the refraction of paraxial rays at a spherical surface, namely,

$$\frac{n'}{u'} = \frac{n}{u} + \frac{n' - n}{r},$$

where u, u' denote the abscissæ, with respect to the vertex A, of the pair of conjugate axial points M, M' ($u = AM$, $u' = AM'$), is immediately apparent. The difference between the two cases is that whereas in the latter formula the abscissæ u, u' and r are measured from the fixed point A, the abscissæ x, x' and e in the former equation are all measured from a variable point E, whose actual position on the axis depends on the slope (θ) of the incident ray. On account of this analogy, Mr. Smith speaks of the point E as "the equivalent vertex" of the spherical refracting surface. The "equivalent radius" e may easily be computed in terms of the angles α, α' and θ, θ' and the actual radius r; thus, for example, in the case shown in the figure, where BE is the bisector of the interior angle at B in the triangle LBL', we may write:

$$EC : BC = \sin \angle EBC : \sin \angle CGB,$$

and since

$$\angle EBC = \frac{\pi - (\alpha + \alpha')}{2} \text{ and } \angle CGB = \frac{\pi + (\theta + \theta')}{2},$$

we obtain:

$$e = \frac{\cos \frac{\alpha + \alpha'}{2}}{\cos \frac{\theta + \theta'}{2}} r.$$

It is also easy to see how the formula which Professor Woodworth published some years ago in a communication to *Science* may be derived from the above. This formula, written in the notation used here, is as follows:

$$\frac{n}{e'} - \frac{n'}{e} = \frac{n' - n}{r} \cdot \frac{\cos \frac{\theta + \theta'}{2}}{\cos \frac{\alpha + \alpha'}{2}}$$

This formula may be quickly derived also (without reference to the abscissa e and the bisector of the angle at B in the triangle LBL') from equation (7) in my paper entitled "Note on Calculation of Path of Ray through a Symmetrical Optical Instrument," published in a previous number of this JOURNAL (iv, 294–299, 1920).

DEPARTMENT OF PHYSICS,
COLUMBIA UNIVERSITY,
NEW YORK, N. Y.,
NOV. 10, 1920.

DIOPTRICS OF THE EYE.

BY

A. AMES, JR., and C. A. PROCTOR.

PREFACE.

THE researches described in the following paper were undertaken for the purpose of determining the exact nature of the image received by the human eye. It was believed that a knowledge of the nature of this image would be of aid in suggesting how the various parts of a picture should be painted to give a pleasing and artistic effect. The steps which led to this belief are as follows:

About 1910 Mr. Ames and his sister, Mrs. Oakes Ames of North Easton, Mass., who were painting together, believed that if they could make an exact reproduction in form and color of a scene they would have a technically satisfactory work of art. To accomplish this they made a very complete set of colors consisting of about thirty-five hundred different colored cards. With these they could match up every shade of color in the scene they were depicting and by making very careful drawings could reproduce very exactly both in color and form any scene they desired. The result, however, from an artistic point of view was not satisfactory. It was in the vernacular "tight" and "hard" and "photographic," and lacked "atmosphere." They were then led to believe that to be technically satisfactory their paintings must be similar to the image received by the human eye.

In the early part of 1913 they painted various pictures introducing in a rough way the distortion which exists in the eye, loss of detail towards the sides of the picture and also a relative increase of blue towards the sides of the picture. Though encouraging, none of these attempts were satisfactory. Believing this lack of success was due to insufficient knowledge of the exact nature of the image the eye receives, Mr. Ames went to Clark University, Worcester, to see the late Dr. J. W. Baird, who was an authority on the physiology of vision. As the information he desired had not yet been ascertained, he went to Worcester to do the necessary research under Dr. Baird early in 1914.

The first problem he undertook was a quantitative determination of the color sensitivity of the peripheral retina; *i.e.*, just what shade a color would appear at various degrees out on the retina. A great many observations were made by a large number of observers. It became obvious from lack of agreement in the readings that certain factors were not being controlled. From various suggestions in the literature [1] and from the fact that a gray spot appeared to the peripheral retina very reddish, it was decided that the uncontrolled factors were the oblique aberrations of the crystalline lens and cornea. Accordingly, early in 1914 a very thorough study of the literature was made and steps taken to determine the aberrations of the human eye. The questions in advanced optics and lens design which arose necessitated the assistance of some one trained in that line. This was first given by a Mr. Herman Beck, then by Dr. A. K. Chapman. The work was stopped when Dr. Chapman and Mr. Ames went into service. After the war, Dr. Chapman went to the Eastman Kodak Company and Mr. Ames went with the work to Dartmouth College, Hanover, N. H., to get the assistance of Dr. C. A. Proctor of the Physics Department, who has collaborated with him in the work since that time.

The false starts, development and discarding of unsatisfactory apparatus and the slow perfecting of the apparatus used, and other vicissitudes common to all research work, were gone through. The work is by no means concluded. It would be nearer the truth to say it has just been begun. Only the eyes of Dr. Chapman and Mr. Ames have been measured and those only for relaxed accommodation, and in one or two meridians. The aberrations determined have been, 1st, Spherical aberrations for different zones; 2nd, Axial chromatic aberration; 3rd, Oblique astigmatism, *i.e.*, the position of the primary and secondary astigmatic fields. It is hoped that the lateral chromatism can also be determined.

It is recognized, of course, that the results of testing one or two eyes cannot be accepted as conclusive. The average readings of many eyes must be found and also under various conditions of accommodation, illuminations, etc.

[1] Especially an article by Pietsch, "Die Ausdehnung des Gesichtfeldes für weisse und farbige Objekte bei Verschiedenen Refraktionszustanden." Breslau, 1896.

It is believed, however, that the description of the apparatus, method of procedure, and such readings as have been obtained are of sufficient interest to warrant their being published at this time.

Special thanks are due to Dr. Baird and to Clark University and its library, which rendered the greatest assistance in bibliographical work, and also to Dr. Chapman. Thanks are also extended to Dr. Kellner of Bausch and Lomb Optical Company, and to individuals in the Eastman Kodak Company.

SPHERICAL ABERRATION.

Spherical aberration is a spreading of rays along the axis near the image plane due to different zones of the lens not having the same focal length.

Considerable work has been done on the determination of the spherical aberration of the human eye. A very good summary is given in Tscherning's "Physiologic Optics," which has been translated into English. Various methods have been used which are described there. It was the purpose of our research to obtain more exact quantitative results than had been gotten heretofore.

To determine quantitatively the spherical aberration of a lens, it is necessary to ascertain the difference in inclination between rays which pass through the axis of the lens and those that pass through zones at various distances from the axis.

Described in general terms, the method used in this investigation was to pass beams of light of small cross section through different zones of the eye and determine the inclination of the beams when their image coincided with the image formed by an axial ray.[2]

DESCRIPTION OF APPARATUS.

The image of a filament in bulb (A), see Fig. 1, after passing through a collimator (B) and prism (C) was focused by lens (D) on a hole in the forty-five degree mirror (E).

Bulb (A) could be adjusted in any position by screws a,a,a, (see photograph, Fig. 3).

The whole system (ABC) could be adjusted in any position

[2] We are indebted to Dr. Kellner for the idea of passing beams of small cross section through the different zones of the eye and the use of strips as described on page 26 to obtain such beams.

by means of the tangent screw at (b) the elevating device at (c) and the tipping screws at (d,d) (see photograph, Fig. 3). The forty-five-degree mirror (E) could also be adjusted in any position by means of the lateral screws at (e,e) the elevating screw at (f) and the tangent screw at (g) (see photograph, Fig. 3).

The light passing through the small hole in (E) strikes a

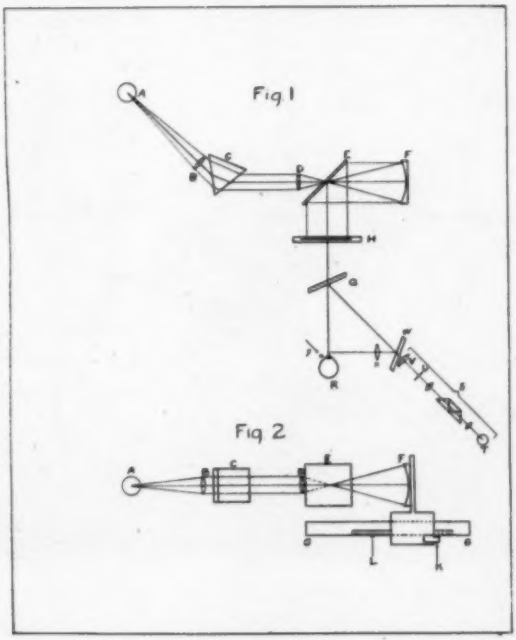


FIG. 1.—Diagram of apparatus, plan.
FIG. 2.—Diagram of apparatus, elevation.

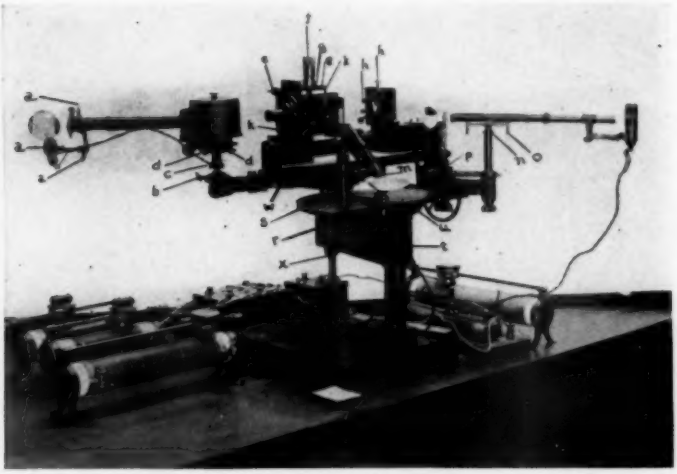
concave mirror (F) which is mounted in a slide that moves back and forth along the bed (GG), Fig. 2. By means of the screws (hh) this mirror can be either tipped or moved laterally (see photograph, Fig. 3).

The light is reflected by this concave mirror (F) back to the forty-five-degree mirror and thence out in the direction (EH). It will be seen that when the mirror (F) is at a distance

from (E) equal to its focal length the light reflected in the direction (E) (H) will be parallel. If it is at a less distance, the light will be divergent; if at a greater distance, convergent. An indicator (K), Fig. 2, attached to (F) gives on the scale (L) marked on the bed the exact position of (F) from which the inclination of the reflected light can be calculated.

At (H) Fig. 1, there is a block with a hole in it, see Fig. 4. In this block two slots are arranged at right angles to each other into which metal strips with slits cut as indicated can be shoved.

FIG. 3.



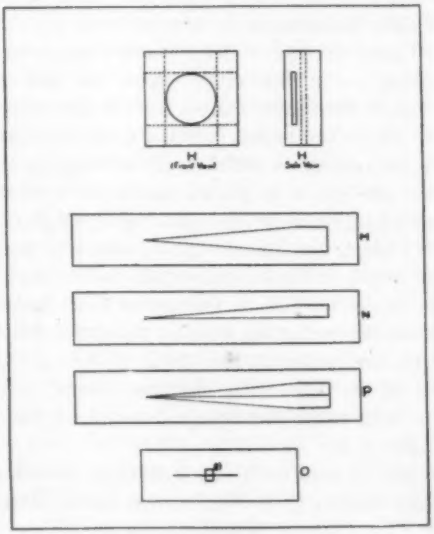
Photograph of apparatus.

This block can be adjusted up or down or sideways by the push screws (k) (k), (see photograph, Fig. 3). It can also be rotated in its mount around an axis perpendicular to the plane in which it lies. The width of the slits in the strips can be adjusted as desired. It was found that if they were too narrow, diffraction bands became too prominent. After trying varying widths slits of .5 mm. in width were used.

It will be seen that if strip (O), Fig. 4, is shoved into block (H) all the light will be shut off except a narrow band which will come through the slit in (O). If strip (M) is then shoved

into block (H) at right angles to strip (O) all the light on the narrow band will be cut off except where the slits in strip (M) coincide with the slit in (O), which will leave two beams whose cross section is determined by the width of the slits in (M) and (O). By shoving (M) forward and back, these two beams will move together or apart as desired. If strip (N) is used, two similar beams are obtained. But upon moving (N) forward and

FIG. 4.



Detailed drawing of block and strips.

back, one of the beams will remain stationary and the other will move relative to it. Similar results are obtained with strip (P).

In this way two or more beams of light of any desired cross section are obtained which can be given any desired inclination to each other, or whose inclination with any setting of the mirror (F) can be determined. Their separation can also be controlled and light of any wave-length can be obtained by changing the position of (A) and (B), Fig. 1, relative to that of the prism (C). We have therefore all the means necessary to determine the

spherical aberration of a lens, granting that the lens can be held stationary. This was one of the most difficult problems to solve in connection with the eye and was accomplished as follows:

On a plane parallel placed at (Q), Fig. 1, a small white speck was placed. The plane parallel is mounted with optical wax on a small table which can be elevated or lowered, moved laterally, and tilted by screws (see photograph, Fig. 3). This speck was illuminated by focusing upon it the image of a small light placed at (m), (see photograph, Fig. 3). This illuminating system is fixed to the small table so that when the white speck is adjusted, the light which illuminates it moves with it. The head was positioned so that the eye of the observer came at (R), Fig. 1, the chin resting on a chin rest. With the eye accommodated for distant vision the speck images behind the retina. It appears on the retina where the retina cuts the cone of light as a circular disk of light, the border of which is determined by the edge of the pupil. When the speck is placed upon the optical axis of the system, a ray of light from (E) passing through the slits in, say (O) and (N) along the axis of the system will pass through the centre of the pupil, if the eye is so positioned that the image of the ray falls in the middle of the above described disk of light. The shadow of the radiating cellular structure of the crystalline lens thrown on the retina by the speck at (Q) assists in centring in the middle of the disk. The slightest lateral movement of the head and eye will cause the image formed by the ray to move across the light disk.

To hold the accommodation constant, which is absolutely imperative, the system (S), Fig. 1, was used. It consisted of a light source (T), Fig. 1, the light from which after passing through a direct vision prism was focused on a vertical line, scratched on a silvered glass disk (u, Fig. 1) which is at the focus of the lens (V), Fig. 1. Monochromatic light was used to make the image sharper and to obviate any inclination to focus for different colors that exist in white light. The system was so placed that with the eye properly positioned the monochromatic line which appeared to be at infinity was visible by reflection from the plane parallel (Q). The system could be raised, lowered and tipped by adjustments (n and o), (see photograph, Fig. 3), and could be turned laterally by a fine tangent screw at (p). The line (U) can be placed at any apparent distance from the eye by

changing its distance from (V). In all readings made to date the apparent position of (U) was at infinity.

For directing rays into the eye obliquely that part of the apparatus designated by (A), (B), (C), (D), (E), (F), (G), and (H) is mounted on an arm, see (r), Fig. 3, which revolves about a vertical axis the centre of which lies directly below the nodal point of the eye when it is positioned. The semicircular plate, see (s) Fig. 3, is marked in degrees and an indicator attached to the above-mentioned arm shows in degrees the obliquity at which the rays strike the eye. A set screw at (t), Fig. 3, enables this movable system to be locked in any desired position.

When rays are put into the eye obliquely it is necessary that the position of the eye be also fixed along the axis of vision, *i.e.*, that there shall be no forward and backward movement of the eye. This is accomplished by putting a half silvered surface at (W), Fig. 1. The monochromatic accommodation line from system (S) will be visible through the half silvered surface. An eye positioned at (R) will see reflected from the front surface of the plane parallel (Q) and the half silvered plate (W) a profile image of itself. A lens (x) is placed at its focal distance from the eye so that this profile image of the eye will be at infinity and can be seen sharply when the eye is focused on the monochromatic line from the system (S). It was found more convenient to put an illuminated spot at (y), Fig 1, close to the nose on the farther side of the cornea from (W). This illuminated spot is seen reflected from the surface of the cornea, appearing like a new moon. The slightest forward or backward movement of the head will cause this moon-shaped image to move across the circle of light formed on the retina by the illuminated speck on (Q).

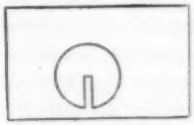
A revolving drum is attached directly under the bed (GG), see (u), Fig. 3. A sheet of paper can be placed around and attached to the drum by a metal strip. The indicator (K) is extended down so that when it is pressed, a knife-edge point will make a mark on the paper.

Before taking readings, the concave mirror (F), Fig. 1, is set to give parallel light and the knife-edge point is lightly pressed on the paper and the drum revolved. This leaves a line on the paper. When readings are made by pressing the knife-edge point into the paper—the drum being turned slightly between each reading—the position of the mirror in front or behind its focal

position is given. This saves considerable time, and makes it unnecessary to turn on a light while making observations.

It was believed that in measuring oblique astigmatism it would be of advantage to have another zonal ray whose inclination could be varied independently of the zonal ray from mirror (F). To accomplish this a mirror similar to (F) was made with a slot cut in it as shown in Fig. 5. This allowed the light from the hole in (E) to pass through it and strike a mirror with a longer focus which travelled on the bed (GG) in the rear of (F). The light was reflected from this rear mirror back through the slot in (F)

FIG. 5.



Detail of front mirror used in obtaining two rays with independently variable inclination.

to the eye. Up to the present no use has been made of this elaboration except in attempts to perfect our methods of measuring spherical aberration where it was not found to be advantageous.

MEASUREMENTS AND DIMENSIONS.

The following more specific measurements and dimensions are given to facilitate, if it is desired, the duplication of the apparatus.

The various parts will be taken up as they are lettered in Figs 1 and 2.

A — 28 volt 25 w. nitrogen bulb with tightly wound filament. A bulb with a double "V" filament was used and black paint put on the glass of the bulb to blank off all but a straight section of the "V." The brightness of the illumination was controlled by a slide resistance 290 ohm. 1. amp.

B — corrected lens 25 cm. focal length.

C — 60° prism; 42 mm. high, 32 mm. across base.

D — corrected lens about 60 mm. focal length.

E — silvered or platinized optical flat 25 mm. in diameter, 2 mm. thick. The hole scratched in silver or platinum was about .03 x .05 mm. Smaller holes were tried but given up on account of objectionable diffraction effects. To prevent light coming through the hole from

internal reflections of the glass plate, all of the back is covered with black balsam paint except a small portion behind the hole in the silver.

F — concave spherical surface of about 72 mm. focal length, silvered or platinized, 24 mm. diameter. The spherical aberration in this mirror at 3 mm. from its axis amounts to only 1/109 diopters, so can be neglected.

G G — bed made of circular steel rod 4 cm. in diameter, with one side flattened off and a mm. scale marked on flat side. The slide on which mirror (F) is mounted is a block turned to fit the rod with an adjustable key to take up wear which fits against flattened side of rod.

H — built up block 45 mm. x 45 mm.
slots to take metal strips 25 mm. x 1.5 mm.
hole in centre 24 mm. in diameter.

M N & P — steel or brass strips 1.5 mm. thick, 25 mm. wide, 15 cm. long. The "Vs" in the centre are adjustable back and forth, which gives control of the width of the slits. The sides of the slits are bevelled back to prevent any chance of reflection from the edges of the metal.

O — same as above except its length is 7.5 cm.

Q — plane parallel 13 mm. wide, 30 mm. long, 3.5 mm. thick, with edges bevelled as shown in (Q), Fig. 1. The white speck was made by putting on the centre of the glass a small drop of white ink and scratching it down to a round disk of about .05 mm. in diameter. This speck is illuminated by focusing upon it the light from a small 2.7 V bulb with a small lens of about 2.5 cm. focal length. The brightness of the light is controlled by a sliding resistance of 67 ohms 2.5 amp.

S — This system was built up of a small telescope with an objective of 20 mm. diameter and about 13 cm. focal length, a silvered or painted disk with a cross made by a long vertical line with a short horizontal cross line, scratched on it in the focus of the objective and a system to give monochromatic light, consisting of a direct vision prism, the necessary lenses and a small 6 volt bulb with a "V" shaped filament, which was mounted so it could be placed in any position. Change in wave-length of image was accomplished by moving the filament. The brightness of the light was controlled by a slide resistance of 67 ohms 2.5 amp.

W — half silvered plate 25 mm. x 25 mm., whose inclination was adjusted by tipping screws, such as are used to control the inclination of a plane parallel in a Michelson Interferometer.

X — a spectacle lens of the proper focal length.

Y — The illuminated spot was produced by focusing the light from a small 6 V bulb into the end of a small solid glass tube which has been silvered on the outside. The end of the tube is ground off at an angle of 45° and polished, and opposite this face the circular side

of the tube is ground down to a flat surface. The light is reflected by the 45° face on this ground surface, the brightness of which is controlled by a slide resistance. The tube is mounted on a stand which is placed on the table (S), Fig. 3, and moved so that the spot of light can be put in the desired position.

ADJUSTING APPARATUS.

First align the apparatus and adjust (A) (B) (C) (D) (E) and (F), Fig. 1, in a rough way so that the eye positioned approximately at (R) will see a yellow light of bright intensity.

Adjust (A) (B) (C) and (D) so that the light when viewed through a spectroscope will be monochromatic. Tip the mirror (F) until the light from it falls upon the hole in (E). This can best be seen by interposing a ground-glass surface between (Q) and the eye. As the mirror (F) is moved back and forth, a round yellow disk will be seen on the ground glass, changing in size according to whether the light is focused on it or in front or behind. When the light is focused approximately on the 45° mirror (E), a yellow disk of considerable size will appear on the ground glass. If any part of the beam is falling on the hole in (E), a dark spot will appear in the yellow disk. By adjusting (F), or if more convenient (E), this spot will fall in the middle of the yellow disk and when (F) is moved to its focal distance from the hole in (E), all the light will pass back through the hole again and the yellow disk will become dark all over. When this takes place, we know that the axis of the mirror (F) and the hole in (E) are in the same line.

Plate (Q) with the small white speck on it is then adjusted so that when the light is focused on the speck the yellow disk on the ground glass becomes dark as described before. This is due to the opacity of the white speck preventing the light from passing any farther. We now know that the hole in (E) the axis of (F) and the white speck on (Q) are all on the same line.

Two strips similar to (O) are then shoved into (H). The only light coming beyond (H) will be that which comes through a square hole the size of the width of the slits in the two strips (O) (O). It will appear as a small square on the ground glass except when the light is focused in the hole itself when it will appear as a small round disk. (H) is then adjusted sidewise and up and down until when (F) is moved forward and backward the light on the ground glass remains stationary and opens out evenly

from a square to a circle and *vice versa*. We then know that the centre of the hole formed by the cross slits (O) (O) is on the axis of the system and that the inclination of the beam of light that comes through it remains the same, no matter what the position of (F).

The system (S) is then adjusted to give an image at infinity of monochromatic light. The slits (O) (O) are removed from (H); a telescope focused for infinity is placed at (R), and (F) is set to give parallel light. In the telescope will be seen a sharp image of the hole in (E) and the image of the line from system (S). System (S) is then adjusted so that the centre of the line falls exactly in the middle of the hole in (E).

The apparatus is now adjusted for taking measurements of spherical or axial chromatic aberration.

Further adjustments have to be made if rays are to strike the eye obliquely.

First, a small rod ground to a fine point, is inserted in a hole centred in the bearing, about which the arm (r), Fig. 3, and the system (A) (B) (C) (D) (E) (F) (H) revolve. This small rod is moved up and down until the top of the fine point is in the plane of the axis of the system (A) (B) (C), etc. This system is then turned about its other bearing (X), Fig. 3, until the focused light falls on the point of the small rod. The system is then revolved to a position at right angles to its former position and adjusted by turning about its bearing (X), Fig. 3, and tipping with screws at (W), Fig. 3, until the light can be again focused on the point of the small rod. By swinging the system (A) (B) (C), etc., to its extreme positions it can be adjusted so that in any position the focused light will fall on the same spot. We know then that this spot is in the centre of the system and that if the nodal point of the eye is placed here an axial ray will always pass through it, no matter what its obliquity.

To arrange (W) (X) and (Y), Fig. 1, so that the nodal point of the eye can be put in this position a thread is hung in front of the fine point on the small rod, at a distance equal to that between the front of the cornea and the nodal point of the eye. This is about 7.07 mm., (W) (X) and (Y) are then so positioned and adjusted that on looking through the telescope we see the shadow of the thread fall exactly upon the vertical line from system (S). With the eye so placed at (R) that the

shine from the surface of the cornea from the light (Y) falls on this line, the nodal point of eye will be at the centre of the system. As we have said before, the slightest forward or backward movement of the eye will cause this shine to move away from that position.

METHOD OF TAKING READINGS OF SPHERICAL ABERRATION.

Various methods of taking readings were tried out.

The apparently simplest one was to use strips (N) and (O), Fig. 4, in the block (H) with a monochromatic light source. These would give an axial ray and another ray which would strike the eye at any desired distance from the axis. When the eye was properly positioned and the mirror (F) set for parallel light,

TABLE I.

Spherical Aberration. (Dr. Chapman.)

Observer, Dr. Chapman. Right eye, vision normal. No corneal astigmatism. Both eyes treated with homatropine, pupil dilated to about 7.5 mm. Accommodation relaxed. Focal length of mirror (F) = 71.9 mm scale reading 16.40. Hole in 45° plate (E) is at 92.1 scale reading. Distance from eye to mirror (F) = 277 mm.

Wave-length	Zone distance from axis, mm	Average scale readings, mm	Calculated object distance in front of eye, V′ mm	Calculated distances behind retina where incident parallel rays cut axis, mm
Red, 6560.....	1.0	167.3	−1361	0.22
	1.5	165.8	−2667	0.12
	2.0	165.1	−4495	0.06
	2.5	166.4	−1949	0.16
	3.0	167.4	−1315	0.22
	3.5	167.5	−1272	0.23
Yellow, 5890.....	1.0	164.0	∞	0.00
	1.5	163.0	+ 5375	−0.05
	2.0	163.5	+10547	−0.02
	2.5	164.3	−17023	0.02
	3.0	166.6	− 1783	0.17
	3.5	166.9	− 1578	0.19
Blue, 4860.....	1.0	161.8	+ 2555	−0.11
	1.5	159.6	+ 1380	−0.20
	2.0	161.1	+ 1988	−0.14
	2.5	162.4	+ 3436	−0.08
	3.0	163.7	+17437	−0.01

For calculation of V′ equation $1/V = 1/71.9 - 1/U$ was used. U = scale reading minus 921, and V′ = 277 − V.

A typical set of readings from which one of the above averages was determined contained sixteen readings. Their average was 16.43. The largest reading was 166.0, the smallest 161.9.

the images from these two rays would not coincide on account of spherical aberration. Mirror (F) could then be moved until they did coincide and the inclination of the ray determined, which would give the spherical aberration for the zone tested.

This method was used by Dr. Chapman in making the readings given in Table I. Though good it lacked accuracy from the fact that the images formed by the rays are relatively large, due to the diffraction bands occasioned by the small dimension of the source and of the holes from the combined slits. The displacement of the two spots on the retina was very evident, but it was difficult to judge when they were exactly superimposed. The holes could not be made larger without making the cross section of the rays too great to measure the small separation in zones which was desired.

It was thought it might be easier to make this exact superimposition if one of the images was one color and the other another. This was accomplished by focusing another source of different color on the hole in (E) by reflection from the face of the prism (C) and putting small pieces of different monochromatic filters over the holes in the slits. This gave slightly better results.

Another method tried was to vary the widths of the slits so that a rectangular instead of a square hole was formed, the long dimension of one hole being at right angles to that of the other. The short dimension was made so small that considerable extension of the image was produced by diffraction. The result was two line images at right angles to each other which when superimposed appeared like a cross. This method was better, and together with different colored spots, was used by Dr. Chapman in observations given in Table II.

The method finally adopted was as follows: Instead of obtaining our axial image from a ray of small cross section passing along the axis, the image of the fine fixation line from system (S), Fig. 1, was used. Its image, which was formed by light coming through the entire pupil, was much smaller than the image formed by the ray of small cross section and enabled much more accurate settings to be made. The centre of the fixation line was carefully adjusted in the middle of the hole in (E) while being observed in a telescope as described on page 33. Strips (M) and (O), Fig. 4, were used with this method. A small

plate, see (I) (O), Fig. 4, was placed on strip (O). By sliding strip (O) this small plate would cut off either side of the "V" in (M) or leave both sides open as desired. This enabled the zones on both sides of the eye to be tested separately or together. Light of only one color was used. The observations taken in this way are given in Tables III, IV, and V.

TABLE II.

Spherical Aberration. (Dr. Chapman.)

Conditions were the same as in Table I except that DR. CHAPMAN'S pupils were not dilated and he used light of different wave-lengths in the two beams, and apertures of slits were ½ x ¼ mm with the long dimension of one at right angles to that of the other.

Wave-length	Zone distance from axis, mm	Average scale readings, mm	Calculated object distance in front of eye, V′ mm	Calculated distances behind retina where incident parallel rays cut axis, mm
Red, 6560	0.50	170.8	− 620	0.48
	0.75	168.6	− 986	0.30
	1.00	167.1	−1530	0.20
	1.50	165.5	−3308	0.09
Yellow, 5890	0.50	166.1	− 2324	0.13
	1.00	164.3	−17090	0.02
	1.50	163.1	+ 5882	−0.04
Blue, 4860	0.50	164.2	−25710	0.02
	1.00	161.8	+ 2488	−0.12
	1.50	159.7	+ 1340	−0.21

$V' = 212 - V$.

A typical set of readings from which one of the above averages was determined contained five readings. Their average was 164.3. The largest reading was 164.7, the smallest 163.7.

Dr. Chapman made the following settings with the slits removed, *i.e.*, with his pupil filled with light. Accommodation relaxed.

Wave-length	Average scale readings, mm	Calculated object distance in front of eye, mm	Calculated distance behind retina where incident parallel rays cut axis, mm
Red, 6560	166.3	−2000	0.144
Yellow, 5890	164.9	−5600	0.052
Blue, 4860	162.8	4600	−0.059

$V' = 277 - V$.

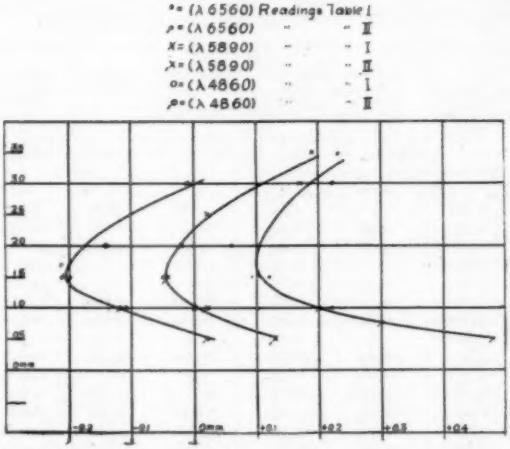
MEASUREMENTS.

The first measurements were made by Dr. Chapman, whose vision is normal, at Clark University, Worcester, Mass., in July,

1917. He measured the differences in aberration of the zones directly above the axis. He used two beams, one on the axis and a movable one above. Instead of a plane parallel at (Q), Fig. 1, he used a very thin cover glass and in place of a monochromatic source he focused on very small point sources at infinity illuminated by white light.

In making the readings given in Table I, he dilated his pupil with homatropine and used beams .5 mm in cross section and the same colored light in both beams. He made the readings given in Table II with normal pupil, using light of different color

FIG. 6.



Spherical aberration curves from measurements of Dr. Chapman's eye. Abscissæ represent 0.1 millimeters from retina. Ordinates represent distance in millimeters of zone from axis.

in the two beams with beams which were narrower in one cross section than in the other, giving the effect of two short lines at right angles to one another.

Fig. 6 shows in diagrammatic form the distances in front and behind the retina at which light of the different wave-lengths is focused by the different zones.

Fig. 10 shows the actual form of the ray bundles due to spherical aberration for the three colors. In making the figures it was assumed that the aberrations on the lower half of the vertical meridian were the same as those on the upper half. This accounts for the symmetry of the bundles.

The displacement along the axis of the curves in Fig. 6 and the bundles in Fig. 10 for different wave-lengths is due to the axial chromatic aberration of the eye which will be treated in more detail later.

The measurements by Mr. Ames were made in 1919 at Dartmouth College, Hanover, N. H. He had considerable trouble getting consistent readings, especially for the zones near the axis.

TABLE III.

Spherical Aberration. (*Mr. Ames.*)

(Horizontal Meridian.)

Observer Mr. Ames, right eye, slight myopia and astigmatism. Pupil normal. Accommodation relaxed. Focal length of mirror (F) = 71.9 mm; scale reading when set for parallel light, 17.62 cm. Distance eye to mirror (F) = 205 mm.

Wave-length	Zone distance from axis, mm	Average readings (distances from focus of mirror), mm		Calculated object (distances in front of eye), mm		Calculated distance behind retina where incident parallel rays cut axis, mm	
		Temp.	Nasal	Temp.	Nasal	Temp.	Nasal
Red,							
6563	0.50	1.71	1.56	− 2800	− 3100	0.103	0.093
	0.75	1.58	1.38	− 3070	− 3570	0.095	0.081
	1.00	1.98	1.03	− 2360	− 4850	0.124	0.059
	1.50	0.71	−0.12	− 7150	∞	0.040	0.000
	2.00	0.39	+0.03	−13100	∞	0.022	0.000
	2.50	0.31	1.05	−16700	− 4750	0.017	0.060
	3.00	0.60	0.87	− 8400	− 5800	0.034	0.049
Yellow,							
5893	0.50	+0.65	+0.40	− 7750	−12750	0.037	0.022
	0.75	1.26	0.12	− 3950	∞	0.073	0.000
	1.00	−0.08	−1.04	∞	5300	0.000	−0.055
	1.50	−1.58	−2.04	3500	2700	−0.083	−0.108
	2.00	−1.05	−1.56	5250	3550	−0.056	−0.080
	2.50	−0.76	−0.79	7070	6850	−0.042	−0.043
	3.00	−0.54	−0.46	9750	11450	−0.031	−0.026
Green,							
5461	0.50	−1.02	−0.78	5400	6900	−0.054	−0.042
	0.75	−0.99	−0.72	5550	7450	−0.052	−0.039
	1.00	−2.39	−2.62	2350	2170	−0.123	−0.134
	1.50	−3.72	−3.95	1600	1510	−0.180	−0.191
	2.00	−3.90	−3.56	1530	1670	−0.188	−0.173
	2.50	−3.16	−2.59	1850	2200	−0.156	−0.132
	3.00	−3.37	−2.68	1750	2130	−0.165	−0.136
Blue,							
4861	0.50	.....	.....	2000	2252	−0.145	−0.128
	0.75	.....	.....	1754	1949	−0.166	−0.149
	1.00	−6.26	−6.72	1035	978	−0.276	−0.292
	1.50	−6.72	−6.73	978	977	−0.292	−0.292
	2.00	−6.20	−5.97	1043	1075	−0.273	−0.266
	2.50	−5.50	−4.85	1147	1273	−0.250	−0.225
	3.00	−5.98	−5.79	1073	1100	−0.267	−0.260

After much experimentation it was found that this was due to failure to get the axis of his eye on the axis of the optical system of the apparatus. This was in turn due to the decentration of his pupil. After the axis of his eye had been determined, his readings showed very little variation. The method of determining this was as follows:

Two beams with the separation of 1 mm. were obtained with slits (M) and (O), Fig. 4. The mirror (F) was pushed slightly in front of its focus position so that the images from the two beams were just separated. A lateral movement of the eye caused the images to move relative to each other. A position was found where movement towards either side caused the images to come together. In this middle position the beams were coming into the

TABLE IV.

Spherical Aberration. (*Mr. Ames.*)

(Vertical Meridian.)

(Conditions the same as in Table III.)

Wave-length	Zone distance from axis, mm	Average readings (distances from focus of mirror), mm		Calculated object distances in front of eye, mm		Calculated distances behind retina where incident parallel rays cut axis, mm	
		Lower	Upper	Lower	Upper	Lower	Upper
Red, 6563	0.5	+1.07	+1.76	−4670	−2730	0.062	0.107
	1.0	−0.53	+0.15	9900	∞	−0.030	0.000
	1.5	−1.03	−1.27	5350	4370	−0.054	−0.067
	2.0	−0.82	−1.56	6580	3540	−0.045	−0.080
	2.5	−1.36	−2.04	4070	2690	−0.072	−0.106
	3.0	−2.12	−3.02	2600	1925	−0.112	−0.151
Yellow, 5893	0.5	−0.53	−0.41	9900	13000	−0.030	−0.022
	1.0	−2.82	−1.32	2040	4200	−0.142	−0.070
	1.5	−3.01	−2.66	1930	2150	−0.150	−0.135
	2.0	−3.31	−4.17	1775	1440	−0.163	−0.200
	2.5	−3.14	−3.57	1860	1660	−0.156	−0.174
	3.0	−4.63	.....	1320	.....	−0.217	.....
Green, 5461	0.5	−1.59	−1.09	3470	5050	−0.084	−0.058
	1.0	−3.22	−3.44	1820	1715	−0.158	−0.168
	1.5	−4.82	−4.82	1275	1275	−0.225	−0.225
	2.0	−4.81	−5.35	1280	1175	−0.224	−0.244
	2.5	−4.92	−5.47	1255	1155	−0.228	−0.248
	3.0	−5.99	.....	1075	.....	−0.266	.....
Blue, 4861	0.5	−5.24	−5.88	1193	1087	−0.240	−0.263
	1.0	−7.05	−6.31	940	1027	−0.303	−0.277
	1.5	−8.61	−8.04	807	848	−0.352	−0.335
	2.0	−9.09	−8.60	775	808	−0.367	−0.352
	2.5	−9.26	−7.70	763	877	−0.373	−0.325

eye at equal distances from the axis and the point half way between is the axis of the lens system. When the eye is so placed that the monochromatic line from system (S), Fig. 1, falls on this point and is carefully kept there, very consistent readings could be obtained.

Table III shows Mr. Ames' readings for the different wave-lengths for the different zones, both temporal and nasal. Table IV shows his readings for the zones in the vertical meridian. Table V shows his readings for the temporal and nasal zones combined and was made as follows:

TABLE V.

Spherical Aberration. (*Mr. Ames.*)

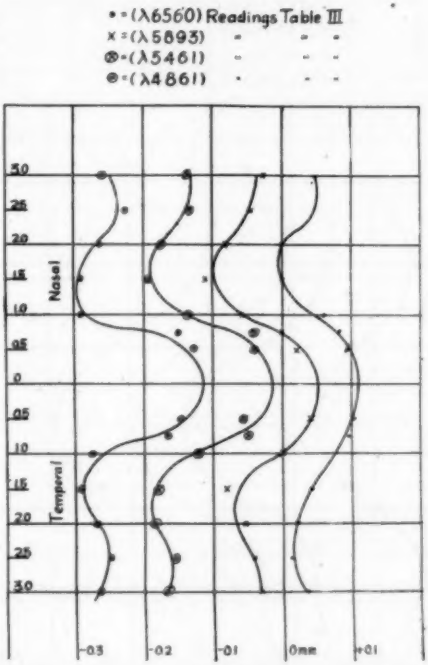
(Horizontal Meridian.)

(Conditions the same as in Table III. Readings for corresponding nasal and temporal zones combined as described on page 41.)

Wave-length	Zone distance from axis, mm	Average readings (distances from focus of mirror), mm	Calculated object distances in front of eye, mm	Calculated distances behind retina where incident parallel rays cut axis, mm
Red, 6563.....	0.50	+0.21	∞	0.000
	0.75	+0.35	−14750	0.018
	1.00	−0.51	10300	−0.028
	1.50	−0.96	5660	−0.051
	2.00	−1.09	5050	−0.058
	2.50	−0.02 nasal only	∞	0.000
Yellow, 5893.....	0.50	−1.08	5080	−0.058
	0.75	−0.81	6660	−0.044
	1.00	−1.61	3420	−0.086
	1.50	−2.60	2190	−0.133
	2.00	−2.35	2390	−0.122
	2.50	−1.27 nasal only	4350	−0.068
Green, 5461.....	0.50	−2.76	2080	−0.140
	0.75	−2.77	2070	−0.140
	1.00	−3.70	1610	−0.180
	1.50	−4.28	1415	−0.023
	2.00	−4.32	1400	−0.205
	2.50	−2.95 nasal only	1960	−0.148
Yellow, 5893 (later reading).....	0.50	−0.50	10500	−0.027
	0.75	−0.85	6400	−0.046
	1.00	−1.75	3150	−0.093
	1.50	−2.66	2130	−0.137
	2.00	−2.49	2270	−0.128
	2.50	−1.78	3090	−0.095

Slits (M) and (O), Fig. 4, were used, both sides of the "V" in (M) being left open. Mirror (F) was moved until both spots superimposed. To insure they were superimposed, slit (O) was pushed back and forward so that plates (1) (O), Fig. 4, cut off first one image and then the other. If one image came exactly

FIG. 7.



Spherical aberration curves from measurement of horizontal meridian of Mr. Ames' eye.

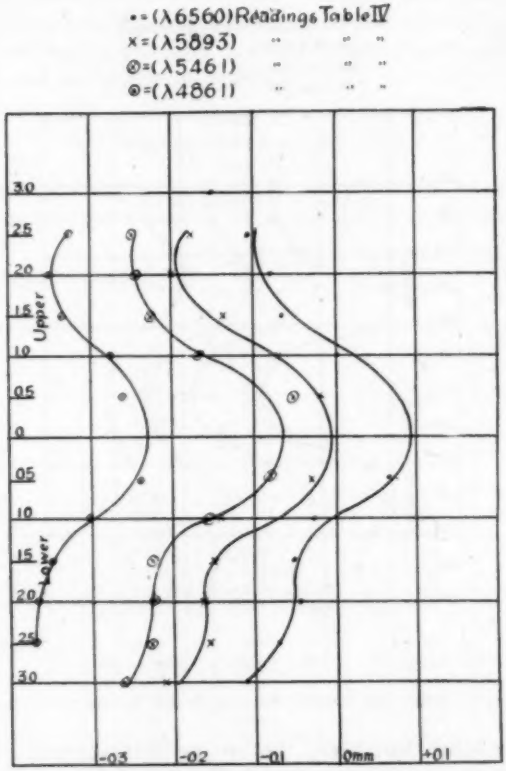
where the other had been, the setting was correct. If there was the slightest jump in one way or the other, a more careful setting was made.

In all readings the wave-length of the vertical monochromatic accommodation line at infinity was (λ6563). Owing to his slight myopia, Mr. Ames was unable to get a sharp image if a light of a shorter wave-length was used.

The curves in Figs. 7, 8, and 9 show these measurements in graphic form.

It will be seen that the part of the curves showing the spherical aberration in the outer zones in the vertical meridian (Fig. 8) are displaced further forward than those in the horizontal merid-

FIG. 8.



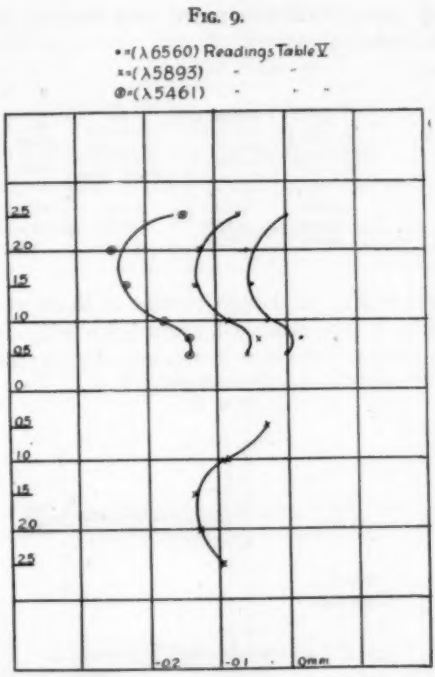
Spherical aberration curves from measurements of vertical meridian of Mr. Ames' eye.

ian (Fig. 7). Difference in shape of the curves in the two meridians is probably due to asymmetry in shape of the cornea similar to that found by Gullstrand[3] caused by corneal astigmatism.

[3] Helmholtz Physiologischen Optik, 3rd Ed., pp. 268 and 269, Vol. 1.

This corneal astigmatism, or the difference in refraction for Mr. Ames' eye in the horizontal and vertical meridian, is shown

FIG. 9.



Spherical aberration curves from measurement of horizontal meridian of Mr. Ames' eye, the aberration from corresponding zones on both sides of the axis being determined at the same time.

by the difference in the position of sharp focus of a vertical and horizontal line source which is as follows:

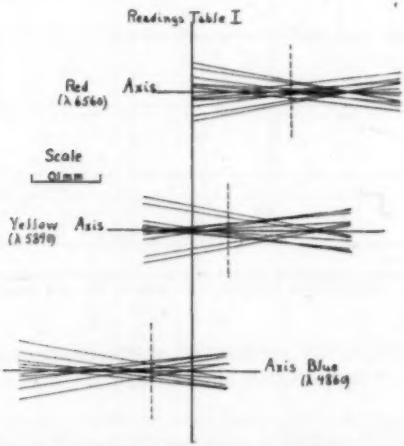
Wave-length	Vertical line, calculated object distance, mm	Horizontal line, calculated object distance, mm	Vertical line, calculated image distance from retina, mm	Horizontal line, calculated image distance from retina, mm	Difference, mm
Red, 6563.....	286000	4030	0.001	0.072	0.071
Yellow, 5893.....	3000	1670	0.095	0.172	0.077
Blue, 4861.....	1100	843	0.260	0.342	0.102

It is also shown in the difference in readings made with a horizontal and vertical beam of light extending across the whole pupil. The following readings were made with strip (O), Fig. 4, placed in a horizontal and vertical position. Accommodation relaxed.

Wave-length	Horizontal slit calculated object distance, mm	Vertical slit calculated object distance, mm	Horizontal slit calculated image distance from retina, mm	Vertical slit calculated image distance from retina, mm.	Difference, mm
Red, 6563	19000	4100	0.015	0.059	0.044
Yellow, 5893	2300	1585	0.125	0.181	0.056
Blue, 4861	1010	822	0.287	0.347	0.060

The curves in Fig. 9, though similar in shape and separation to those in Fig. 7, are displaced toward the left. It is believed this is due to a slight difference in accommodation; these measure-

FIG. 10.



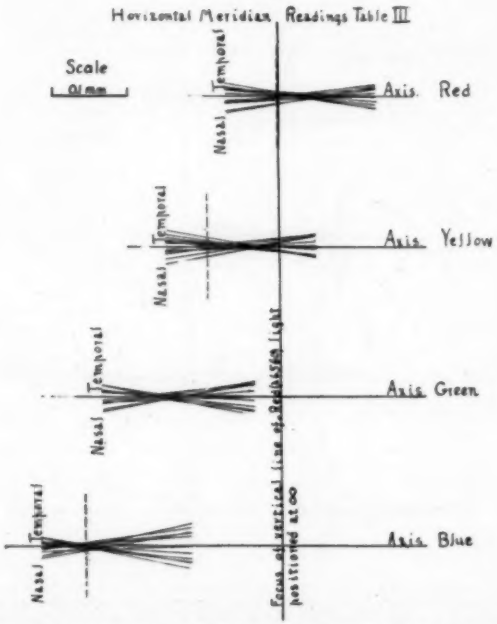
Actual shape of spherical aberration bundles as constructed from measurements of Dr. Chapman's eye. The solid line shows the focus of a spot of white light positioned at infinity. The dash lines show the position of the retina in the bundles when the eye focused on infinity sees most sharply a spot of the given color.

ments being taken about six months later than those shown in Fig. 7.

Figs. 10 (a), (b), (c) show the actual form of the ray

bundles due to spherical aberration. Fig. 10 (a) shows those formed by the horizontal meridian, Fig. 10 (b), those by the vertical meridian and Fig. 10 (c), those formed by the combined horizontal and vertical meridian. It will be noted that the bundles for the different colors in the horizontal meridian are very similar to each other and also that those in the vertical meridian,

FIG. 10a.



Actual shape of spherical aberration bundles as constructed from measurements in the horizontal meridian of Mr. Ames' eye. The dash lines show the position of the retina in the bundles when the eye focused on infinity sees most sharply a vertical line of the given color.

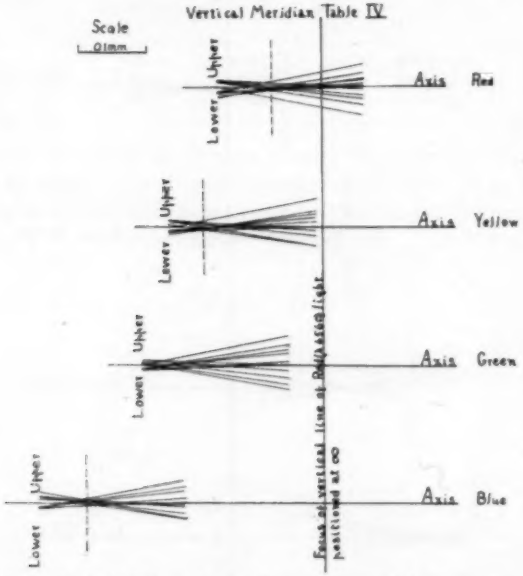
while similar to each other, are different from those in the horizontal meridian. The displacement of the bundles formed by the two meridians is due to the corneal astigmatism which has been mentioned.

It will be seen that while both Dr. Chapman's and Mr. Ames' curves are in general form similar, Dr. Chapman's curves differ from Mr. Ames'. That the spherical aberration should be dif-

ferent in the eyes of different individuals is in conformity with the measurements made by other methods, see Tscherning's "Physiological Optics" referred to above.

The results in general form are similar to the results found by other methods. The aberration is, however, considerably less than that calculated by Gullstrand, see p. 363, Vol. I, Helmholz, "Phy. Opti.," 3rd Ed.

FIG. 10b.



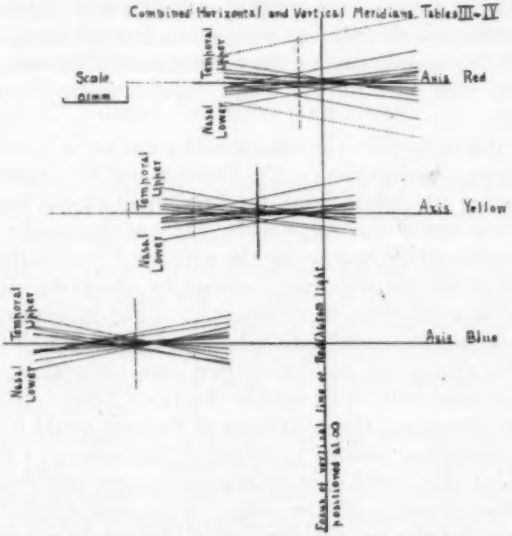
Actual shape of spherical aberration bundles as constructed from measurements in the vertical meridian of Mr. Ames' eye. The dash lines show the position of the retina in the bundles when the eye focused on infinity sees most sharply a horizontal line of the given color.

There is a much mooted question as to whether the eye so focuses that that part of the cone which has the smallest diameter falls upon the retina or that part where the section of the caustic due to spherical aberration is smallest. Both Tscherning and Gullstrand believe that it is that part of the bundle where the caustic is smallest and not where the cone has the least diameter. The exact nature of the results obtained in this method of research make it possible to answer this question very definitely. For

Dr. Chapman's and Mr. Ames' eyes there seems to be no question but that they focus approximately on that part of the bundle where the cone has the least diameter.

The long vertical line in Fig. 10 shows the position of Dr. Chapman's retina when he saw most sharply a white source positioned at infinity. The long vertical lines in Figs. 10a, 10b, and 10c show the position of Mr. Ames' retina when he saw most

FIG. 10c.



Actual shape of spherical aberration bundles as constructed from the combined measurements in the horizontal and vertical meridians of Mr. Ames' eye. The dash lines show the position of the retina in the bundles when the eye focused on infinity sees most sharply a spot of the given color. The dotted lines show the outside rays of a blue (λ4860) bundle when the eye sees most sharply a red (λ6560) spot.

sharply a vertical red line positioned at infinity. The bundles show the position of the image cones of point sources of different wave-length positioned at infinity, their distance from the vertical lines giving the chromatic aberration. The short vertical broken lines show the position of the image for the corresponding wave-lengths of an object placed at infinity. In the case of Fig. 10 this object was a point source; 10a a vertical line; 10b a horizontal line; 10c a point source. These positions are given for Dr. Chap-

man in the tables on page 36, for Mr. Ames in the tables pages 43 and 50.

Figs. 10, a, b, and c, show that the retina shifts its position in the image bundle according to whether Mr. Ames is looking at a vertical or horizontal line or a point source. This is to be expected in view of the difference in the spherical aberration curve in the horizontal and vertical meridian. The sharpness of an image from a vertical line source is in the main governed by the aberrations in the horizontal meridian, that from a horizontal line source by those of the vertical meridian and that of a point by a combination of both. It will be noted in all cases, however, that the retina as indicated by the vertical dash lines falls approximately on that part of the image bundle where its diameter is smallest.

That this is usually the case would seem to be confirmed by the following observations by Dr. Proctor and Mr. Ames. With light entering the whole pupil the brightness of the source was made so low that it was just visible. It was then made to focus as sharply as possible by moving the mirror (F) while the accommodation of the eye was kept constant by observing the monochromatic fixation cross in system (S). The brightness of the source was then increased until it became bright enough to give a glare. No change in accommodation was perceptible, *i.e.*, the fixation line continued to be equally sharp and clear. Nor at any time while increasing the brightness of the spot could it be made to appear sharper or smaller by changing the focus of (F). It is believed that this could only be true if the eye was focused on the smallest diameter of the cone. If it was focused on the smallest part of the caustic, there would be a halo around it, invisible with very low illumination but very bright with high. This halo would be larger than the smallest diameter of the cone and with a very bright source it would be expected that a smaller image could be obtained by a slight shift in focus which would bring that part of the cone which has the smallest diameter upon the pupil.

In view of the definite results it seemed possible that a lens could be made which would correct both the spherical and chromatic aberration resulting in a smaller image bundle. Such a lens is being designed. While it is realized it could not improve vision in an eye like Mr. Ames' where the corneal astigmatism is

relatively large, it might correct the aberrations in one meridian. In an eye with no corneal astigmatism it might possibly result in improved vision.

As has been stated, the readings that have been made are scarcely more than preliminary. The spherical aberration of a large number of eyes should be measured. This should be done for near and intermediate and distant accommodation. The measurements should also be made over at least four different meridians of the eye. The angle between the visual and optical axis (angle alpha) of each observer should be determined as well as the corneal astigmatism and the relationship between these factors and the spherical aberration noted.

AXIAL CHROMATIC ABERRATION.

Introductory.

The literature on axial chromatism of the human eye is very voluminous. Tscherning gives a short chapter to it in his "Physiologic Optics."

One method to determine quantitatively the axial chromatism of a lens is to ascertain the difference in inclination of rays of varying wave-length which come to the same focus. The chromatism may be determined for the lens as a whole by passing a large bundle through its full aperture, or for its various zones by passing small bundles through them.

The first method is, as far as is known, the only one that has been used up to date in measuring the eye. Nutting [4] was the last to use this method and found the mean axial chromatism for seven eyes between the C and F lines to amount to 0.99 D. Young estimated it to be 1.3 D. Fraunhofer found it 1.5 to 3 D for the range of the visible spectrum.

Method of Making Measurements.

The apparatus above described is admirably fitted to make such measurements. The method is as follows: With the slits removed from (H), Fig. 1, so that the whole pupil will be filled with light, the eye is accommodated on the monochromatic fixation line in system (S), Fig. 1. Monochromatic light of a given wave-length is focused through the hole in (E), Fig. 1. Mirror

[4] Proceedings of the Royal Society A, Vol. 90 (1914).

(F), Fig. 1, is then moved until the sharpest image is formed on the retina. The reading will give the inclination of the rays necessary for light of that wave-length to come to focus on the retina. The wave-length of the light coming through the hole in (E) is then changed and a new reading taken. In this way the axial chromatic aberration curve for the different wave-lengths can be determined.

The table, page 36, gives the curve at three points for Dr. Chapman's eye; those on pages 43 and 44 for three points for Mr. Ames' eye. The amount of the aberrations varying from 0.72 D to 0.98 D correspond with the figures found by others as given above.

The method followed by Nutting is practically the same as given above, except that he did not use a monochromatic accommodation control. He also went further in using a greater number of different wave-lengths which enabled him to plot more accurately the entire curve. He finds this curve in some cases considerably flattened through its central portion, *i.e.,* on both sides of the D line and assumes that this is due to the eye being more or less corrected.

TABLE VI.

Chromatic Aberration, Accommodation Relaxed. Monochromatic Fixation.

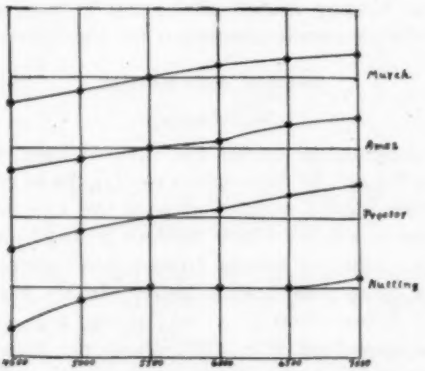
Wave-length	Mr. Ames Fixation, λ=6560		Dr. Proctor Fixation, λ=5600		Mr. Murch Fixation, λ=6560	
	Average scale readings	Retina to focus, mm	Average scale readings	Retina to focus, mm	Average scale readings	Retina to focus, mm
7000	+0.04	0.000	3.46	−0.226	− 2.24	0.127
6500	−0.71	0.038	2.30	−0.147	− 3.60	0.174
6000	−1.55	0.081	0.64	−0.037	− 6.70	0.291
5500	−3.5	0.170	−0.31	+0.017	− 8.30	0.345
5000	−5.8	0.260	−1.90	+0.103	−11.20	0.425
4650	−8.45	0.347	−4.81	+0.248	−14.70	0.508

Dr. Proctor, Mr. Murch and Mr. Ames made readings for a similar set of wave-lengths, the results of which are given in Table VI and Fig. 11. It will be seen that they did not find any flattening of the curve. It is suggested that the flattening found by Nutting was due to his using a chromatic, *i.e.,* white light accommodation object. With an object illuminated by white light the eye normally accommodates so that the yellow light which is of the greatest intensity focuses on the retina. A slight

change of accommodation which would cause a slightly more reddish or more greenish light to focus on the retina would not perceptibly change the visibility of the object. There would be a tendency to make such a shift where the eye was trying to focus a monochromatic source of a wave-length near the D line. This would explain the flattening in this region which Nutting found. The fact that neither Dr. Proctor or Mr. Ames or Mr. Murch found such a flattening when they used a monochromatic accommodation object is confirmation of this.

The determination of the axial chromatism for different zones

FIG. 11.



Chromatic aberration curves for eyes of four individuals; ordinates are relative focal distances, $\delta v/v$.

was found when the spherical aberration of the eyes for different colors was measured. It is shown in Tables I, II, III, IV, and V and in Figs. 6, 7, 8, 9, 10, and 10 (a), (b), and (c). In making the observations shown in Table I and Fig. 6, Dr. Chapman used an accommodation object illuminated by white light. But in making the readings given in Table II and Fig. 6, his axial ray was of one wave-length and his zonal ray of another. With a blue zonal ray he used a yellow axial ray; with a yellow zonal ray, a red axial ray; with a red zonal ray, a green axial ray.

Figs. 10 and 10 (a), (b), and (c) give a very clear idea of the chromatic aberration in the eye. They also give a very definite measure of its size. In Fig. 10 (c), top figure, the dotted

lines represent the outside rays of the blue bundle when the eye is focused on red. The question of the actual size of spherical and chromatic aberration image bundles is of considerable interest. It is our intention to treat the matter in a later paper.

As was said in respect to spherical aberration the work that has been done in axial chromatism is hardly more than preliminary. Readings should be taken on a large number of eyes with different accommodation and with dilated and normal pupil. And measurements should be made with a large number of different wave-lengths.

Two special problems of interest are: 1st, a more positive determination of whether or not a flattening of the chromatic curve, such as Nutting found, exists; 2nd, a more definite determination of the chromatic aberration for the different zones.

OBLIQUE ASTIGMATISM.

Introductory.

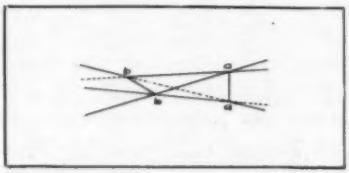
Oblique astigmatism causes the rays of light from a point source not on the axis to focus into a ray bundle of complex form. The ray bundle from a point source on the axis focuses in the form of a cone to a point where the rays cross to spread out into another cone. The ray bundle from a point source not on the axis forms a more complicated figure. In its pure form in a simple lens it focuses first to a line, see aa, Fig. 12, which lies in a position tangential to a circle about the axis of the lens. It then crosses and narrows in its long dimension and lengthens in its short one until it becomes a line again, see bb, Fig. 12, which is perpendicular to the first line.

The image of every point source not on the axis has this peculiar form. The farther the source from the axis the greater the separation between the two parts of the image which have the form of lines. If a sensitive plate or ground glass screen is placed behind the lens the form of the image that is apparent depends upon the position of the screen. If it is placed far back, *i.e.*, to the left of bb, Fig. 12, the image will be in the form of a radial oval; if at bb, in the form of a radial line; if half way between aa and bb, the form of a circle; if at aa, the form of a tangential line; if still nearer the lens, in the form of a tangential oval. If the screen is held stationary relative to the lens, a similar change in form of image can be noted by moving the point source, while

keeping it at the same angular obliquity, from a distance to a position near the lens.

If instead of a point a line source is used, a similar imaging takes place. Every point on the line source is stretched tangentially or radially, depending upon the position of the line source. It can be seen that if the stretching of the various parts of the line source are in the same direction as that of the line source itself, its image will appear perfectly sharp and slightly elongated. That is, if a line source is put in a position tangential to the axis, its image will be sharp when the source is so positioned that the part of the image that forms a tangential line, *i.e.*, aa, Fig. 12,

FIG. 12.



Diagrammatic form of astigmatic image bundle.

falls on the screen. If it is put in a position radial to the axis, its image will be sharp when that part of its image that forms a radial line falls on the screen.

In all simple lenses this characteristic image formation is more or less confused by coma, a one-sided blur.

Bibliographical.

Oblique astigmatism has been recognized to exist in the human eye for a long time. As far as can be found, however, no comprehensive collection of the literature upon the subject has been made. A list and brief summary of all important articles that could be found will be given.

(1) Young, in 1801, remarks on the small extent of the angle of perfect vision. Imperfections which begin a degree or two from the axis are due, he claims, partly to aberration of oblique rays but primarily to insensibility of the retina. He calculates the position of the primary and secondary astigmatic image fields of a circle ten inches from the eye. He uses a homo-

geneous lens and considers rays passing through its anterior vertex. He gives curves showing first and second astigmatic image fields formed by cornea alone, by cornea and anterior surface of lens, and by cornea and whole lens. In each case he shows a curve of the circles of least confusion and says this curve would fall on the retina if there existed a diminution of density in lateral parts of the lens.

Astigmatism, Oblique Rays, Eye.

Investigators	Date	Nature of work	Reference
(1) Young	1801	Theo.	Phil. Trans. 1801, Vol. XCIII, p. 23.
(2) Stammerhaus	1874	Exp.	Arch. für Ophth. Vol. XX, p. 146.
(3) Peschel	1878	Exp.	Pflüg Arch. Vol. XVIII, p. 504.
(4) Hermann	1878–9	Theo.	Pflüg Arch. Vol. XVIII, p. 443.
(5) Fick	1879	Theo.	Pflüg Arch. Vol. XIX, p. 145.
(6) Matthiessen	1879	Theo.	Pflüg Arch. Vol. XIX, p. 481.
(7) Rasmus & Wauer	1879	Theo.	Pflüg Arch. Vol. XX, p. 264.
(8) Matthiessen	1879	Theo.	Arch. für Ophth. Vol. XXV, p. 257.
(9) Peschel	1879	Exp.	Pflüg Arch. Vol. XX, p. 338.
(10) Hermann	1879	Theo.	Pflüg Arch. Vol. XX, p. 370.
(11) Schoen	1879	Theo.& Exp.	Du Bois-Raymond, Arch. für Physiol., p. 146.
(12) Hermann	1882	Theo.	Pflüg Arch. Vol. XXVII, p. 291.
(13) Parent	1882	Exp.	Rec. D'Ophthal., 1882, p. 216.
(14) Schoen	1884	Theo.	Beitrage zur Dioptick des Auges, Leipzig, Engleman, 1884.
(15) Matthiessen	1884	Exp.	Arch. für Ophthal. Vol. XXX, p. 91.
(16) Schmidt Rimpler	1884	Exp.	Congres International des Sciences Medicals, Compte Rendue, III, p. 73.
(17) Albini	1886	Exp.	Giornale de R. Acad. di Torino, 49, p. 657.
(18) Staub	1887	Exp.	Graefe Arch. für Ophthal, 33, p. 84.
(19) Wertheim	1894	Exp.	Zeit f. Psy. & Phy. der Sinn, 7, p. 172.
(20) Heinrich	1895	Exp.	Zeit f. Psy. & Phy. der Sinn, 9, p. 353.
(21) Heinrich	1896	Exp.	Zeit f. Psy. & Phy. der Sinn, 11, p. 410.
(22) Drualt	1900	Exp.	Arch. d'Ophth, 20, p. 21.
(23) Loria	1906	Exp.	Zeit f. Psy. & Phy. der Sinn, 40,p.160.
(24) Heinrich	1909	Exp.	Brit. Journ. Psy., Vol. 3, p. 66.

(2) Stammerhaus, in 1874, was the first to attempt to determine experimentally the position of the astigmatic fields. He used an ophthalmoscope and observed radial and tangential blood-vessels in the retina and determined the aberration by the strength of lens necessary to give a clear image.

In normal eyes he found radial blood-vessels (secondary astigmatic focus) were clearly visible out to about 43°. From there on the retina becomes hypermetropic until at a distance of about 54° it becomes so to the extent of 3.71 diopters. In most eyes

the change took place gradually but in some suddenly. At about 54° there is always hypermetropia of not less than 2.46 diopters.

He found that with tangential blood-vessels (primary astigmatic focus) he could see them clearly with the same lens out to the equator. He also gives measurements of abnormal cases. With a case of central hypermetropia of 3.5 D he found the hypermetropia at periphery increased very little. With a case of central myopia of 6 D at 50° he found the radial vessels 1.5 D myopic, which would show that as the axial diameter increased the other diameters increased also. With another case of central myopia of 5 D at 50° radial vessels were 3.5 D hypermetropic, which would show that the bulbus was stretched in its axial diameter.

(3) and (9) Peschel, in 1878, was the first to try to use radial and tangential lines outside of the eye to determine the position of the astigmatic fields. Unfortunately his observations were limited to his own left eye which was myopic 6.6 diopters. He used black lines on white paper, and placing them first in a tangential and then radial position, moved them back and forth until the lines appeared clear. He measured from 15° out and found that the secondary astigmatic field fell on the retina, the primary field falling in front of it. His results are inconsistent with those of Stammerhaus. This could be explained if his myopia was of the kind measured by Stammerhaus and mentioned above as the second abnormal case. In 1879 Peschel (9) contributed measurements on the extirpated eyes of animals and man. He used the method employed in testing a lens on a lens bench. At 70° he found the distance between the first and second astigmatic image to be 1.5 mm.

(4), (10), and (12) Hermann, in 1878 and 1879 and 1882, published long papers in which he discusses at great length the refraction of rays of oblique incidence with special reference to the eye, taking consideration of its laminated structure. These articles are so technical that no attempt will be made to summarize them.

(5) Fick, in 1879, using Hermann's formulas, figured out the position of a primary and secondary astigmatic image and found the retina near the secondary focus. The contribution is purely theoretical and is questioned by Matthiessen in his articles in the *Arch. für Ophth.*, Vol. 25.

(6), (8), and (15) Matthiessen, in 1879, in an article entitled "The Differential Equations of the Dioptrics of the Human Eye," gives an elaborate theoretical dissertation on the subject. In this same year he published another article entitled "Uber der Geometrische Gestalt der theoretical Retina des periskopischen schematischen Auges," (8) in which he says the work of Rasmus and Wauer and his own (see (6)) show that the retina lies between the primary and secondary astigmatic image fields and criticizes Fick (5). In 1884 (15) he investigates the position in which black letters on a white screen must be placed to focus on the peripheral retina but does not get any very conclusive results.

(7) Rasmus and Wauer, in 1879, were the first to calculate the position of the retina relative to the primary and secondary astigmatic image fields for more than one angle. They made calculations for every 10° from 0° to 90° for both the accommodated and unaccommodated eyes of Helmholtz. They find the retina lies approximately half way between the two fields. They mention Stammerhaus, Schoen, Peschel, Fick and Matthiessen. Their charts for both the accommodated and unaccommodated eye with the retina as calculated by Matthiessen are shown in Fig. 15.

(11) and (14) Schoen, in 1879, investigated the position of the retina relative to the primary and secondary astigmatic image fields at 60°, by observing in an eye dilated with atrophine the image of radial and tangential lines which were thrown on the retina. The apparatus is described in another article by him in *Arch. für Ophth.*, xxiv, 4, p. 91, 1877. He found that in ten out of sixteen cases the retina lay between the primary and secondary image fields. The average distance between the fields was 2.85 D. He does not mention Stammerhaus, though their results agree. He shows that Peschel got 2.1 D between the two image fields. He alsó gives some calculated results which agree closely with those of Matthiessen. In his book (14) Schoen deals at some length with the theory of formation of images by oblique rays.

(13) Parent, in 1882, attempted to determine the position of the primary and secondary astigmatic fields by means of the ophthalmoscope but found it was difficult to hold the ophthalmoscope and the observed person sufficiently still and to find radial and tangential blood-vessels to observe. He, therefore, tried the

skiascopic method, using the keratoscope of M. Cuignet. The refraction for the primary and secondary fields for an emmetropic eye are given in a table and figure in the summary.

(16) and (18) Schmidt Rimpler, in 1884, and M. Staub, in 1887, investigated the difference in refraction between the fovea and papilla (blind spot). Schmidt Rimpler found that with marked ametropia it varied greatly, *i.e.*, from 1 D to 5 D, where the subject had myopia of 5 D. Staub found the average difference about 1.5 D. The ophthalmatic method was used.

(17) Albini, in 1886, believed that the poor vision in the peripheral parts of the retina was due to the aberration of oblique rays. His method of attempting to correct this was by interposing lenses of different strength in the path of the oblique ray until the paraxial object under observation became most evident. He used black spots on white background and colored objects. He found that by interposing the proper lenses his objects could be seen much further out on the periphery, and that all the colors could be made equally perceptible almost out to the full field of vision. Nothing in his work shows that he was cognizant of the nature of the image formed by oblique rays. An analysis of his results, however, shows that he was probably causing the secondary astigmatic image field to focus on the retina. From the strength of the lenses he used the position of this field was calculated and is shown in Fig. 13.

Although mentioned by Baird in "The Color Sensitivity of the Peripheral Retina," Carnegie Inst., May, 1905, little consideration has been given Albini's work by researchers in the color sensitivity of the peripheral retina. The possible importance of it will be discussed in our conclusion.

(19) Wertheim, in 1894, attempts to test the visual acuity of different parts of the retina with a wire grating which could be placed in a radial and tangential position. He could find no evidence of astigmatism on the peripheral retina. He mentions Hermann, Matthiessen and Albini.

(20), (21) and (24) Heinrich, in 1895, 1896 and 1909, makes three most important contributions. In 1895 he investigated the changing of curvature of the lens and variation of pupillary opening under different conditions of attention. He used an ophthalmometer to notice changes of curvature in the crystalline lens and then fixed attention on the peripheral object.

He found that the size of the pupil increased as attention object was moved outward up to 40° or 60°; it then decreased. The curvature of the lens also changed, becoming flatter, *i.e.*, hypermetropic. He mentions oblique astigmatism and raises question as to which image falls on the retina. He also treats in this article of the effect on pupil and lens of giving the attention to cumputing.

In 1896 he investigated the amount of peripheral accommodation that is possible. He found that accommodation for a paraxial object depended on its distance away and on the distance of the central fixation object. He finds it not exact, only approximate. He considers the primary and secondary astigmatic image fields for oblique rays. He speaks of the great importance from a physiological standpoint of determining where the retina lies relative to primary and secondary astigmatic fields; size of focal lines; size of diffusion circle or ellipse. He mentions Hermann, Rasmus, Wauer, Peschel, Schoen, Parent and Butz and discusses their findings. He concludes that

1. Eye has power to accommodate on paraxial objects.
2. Such accommodation depends upon location of axial fixation point.
3. Such accommodation causes coincidence of primary astigmatic image plane with retina.
4. Depth of such paraxial accommodation is smaller than axial accommodation and decreases up to 40° or 60° and then increases.
5. Length of astigmatic focal lines increases with angle of incidence.
6. Length of astigmatic focal lines increases with increase of radius of curvature.
7. The angle at which the refracted rays converge decreases with the increasing radius of curvature and increases with increasing angle of incidence. The first fact favors paraxial accommodation.

His contribution in 1909 (24) will be taken up after Loria's work (23) has been considered.

(22) Drault, in 1900, determined the position of the primary and secondary astigmatic field by skiascopy, the method used by Parent (13) and whose work he mentions. He made readings for every 10° on both nasal and temporal side of lens out to 50°. He gives the average result of twenty-two observations. He says the correction for hypermetropic eyes was from +2.5 to +3.4 D in the vertical meridian. The readings of the abnormal eyes are included in his average, and he gives no average for normal eyes. His results are shown in a table and figure in the summary.

He gives size and shape of the images he found on different parts of the retina, stating it is round or oblong according as to whether it is between or nearer one or the other of the astigmatic fields.

He concludes that

1. The astigmatism of rays in the eye "is a little less than would be calculated."

2. That it is a little less for the temporal part of the visual field than the nasal, due to the inclination of the visual and optical axis.

3. That the retina is situated between the two fields.

(23) Loria, in 1906, gives a most valuable contribution. He reviews Heinrich's work and determines experimentally, 1st, the extent of the field in which tangential lines can be focused on the retina without keeping the axial accommodation fixed; 2nd, the shape of the field in which tangential lines appear most sharp when the axial accommodation is kept fixed. He did his work most thoroughly, taking a great many observations and using the atrophined and normal eye. He used for a stimulus parallel lines 2 mm. wide, 2 mm. apart and 2 cm. long. The shape of the fields which he found with a fixed accommodation are the position in space of the primary astigmatic object field with the given accommodation. The position of the corresponding image fields was calculated and is shown in Fig. 14. Loria was the first to expound the theory that with a given accommodation there is a field in space upon which all points are seen distinctly.

He concludes

1. The accommodation of the eye for paraxial vision is determined by the distance of the stimuli and is independent of the distance of the axial fixation point.

2. The eye is paraxially myopic, increasing with the paraxial angle.

3. The paraxial accommodation zone decreases with increasing angle.

4. It is possible to determine with a fixed curvature of lens a surface in space where the conjugate focus of the retina lies. The surfaces corresponding with the least and greatest curvature of the lens are the limits of the paraxial accommodation space.

5. Objects placed on this surface will be seen most distinctly.

Though working with the primary astigmatic image and object fields, he makes no mention of that fact which he probably did not recognize, nor does he make any mention of the secondary astigmatic image and object fields.

(24) Heinrich, in 1909, investigated a fact noticed in Loria's

experiments, that, when objects lay in the primary astigmatic object field, they all appeared to the observer to be the same distance away. He used fine threads .3 mm. thick in a vertical position and found the positions in space out to an angular distance of 5° where the paraxial thread appeared to be at the same distance as the axial thread upon which the eye was fixed. He obtained very definite readings which he shows in graphs of the object fields. The image field corresponding to the object field shown in Fig. 1, page 70, is shown in Fig. 15. When Mr. Kurtz, the observer, who was myopic 2.5 D, fixated an axial point 60 and 80 cm. distant, a periodical displacement of the equidistant areas took place. Mr. Kurtz described these oscillations in an article following on page 75 of the *British Journal of Psychology*, Vol. 3. Heinrich suggests no explanation. He states that a series of similar experiments were begun on other individuals which are not described. He also states that if his curves are prolonged, they will fit into the curves found by Loria. He concludes that monocular space perception is tri-dimensional. He says on page 74,

"Points in space, which in a given state of accommodation of the lens in monocular vision form well-defined retinal images, represent a surface which serves as a fundamental surface in relation to which the position of other points in the third dimension is determined. The points outside such a surface are regarded as more distant, those within it as lying nearer.

"The images of the points outside the surface are diffusion circles. With these diffusion circles such factors as the degree of diffusion, light-intensity, etc., will become of importance; and these factors contribute to the appearance of depth. In other language, the image formed on the retina by means of the human dioptric apparatus must include all those factors which correspond to the report of depth. An analogy may make this clearer. It is well known that, in a picture representing objects in spatial order, every object must have its local tone. It is this local tone, and not the geometrical perspective, which determines the position of the object in space. Any object not in the projected line does not fit into the space of the picture. Something analogous may be supposed to occur in the case of the image formed on the retina. Yet another example. The images of objects formed by means of a badly constructed system of lenses appear on a screen as ill-defined. Yet the whole image is characterized by an exaggerated effect of depth."

It seems without question that he was determining the position of the primary astigmatic field, although he does not mention the fact. Nor does he make any suggestion of trying to determine the position of the secondary astigmatic field in a similar manner.

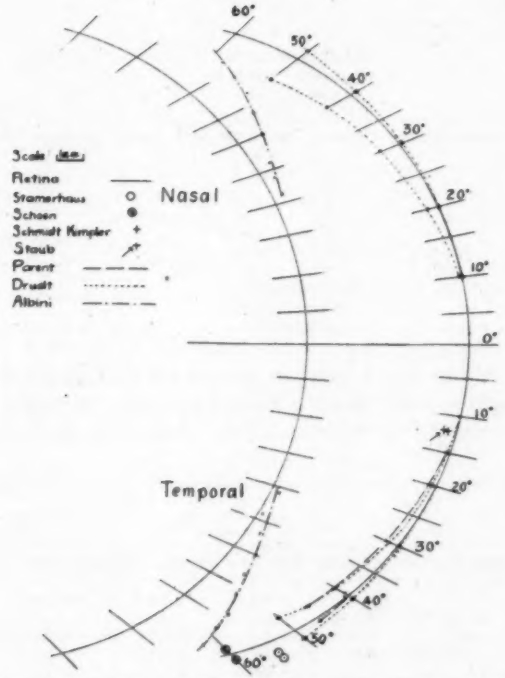
SUMMARY.

The results of the experimental work that has been done are on the whole very consistent.

Three different methods were used:

1st. Ophthalmoscopic—Stammerhaus, Schoen, Schmidt Rimpler, Staub.

FIG. 13.



Position of the primary and secondary astigmatic image fields relative to retina as computed from data by authors indicated.

2nd. Skiascopic—Parent, Drualt.

3rd. Subjective determination of position of sharp focus of paraxial stimuli—Peschel, Albini, Wertheim, Heinrich, Loria.

In consideration of the results due allowance has to be made for the distorted shape of the bulbus which is apt to be found in ametropic cases.

The result of the ophthalmoscopic measurements are put in the following table for comparison:

	Stammerhaus		Schoen		Schmidt Rimpler		Staub	
Angle	Prim. field	Second. field	Prim. field	Second. field	Prim. field	Second. field	Prim. field	Second. field
12.5°	0	0	.....	.....	−1.0 to −1.5D	0	−1.5D	0
43°	0	0	.....	.....	.....	.....	.....	.....
54°	0	+2.46 D to +3.71 D	.....	.....	.....	.....	.....	.....
60°	0	.....	Difference between fields = 2.85 D		.....	.....	.....	.....
Calculated Corresponding Primary and Secondary Image Distances from Retina in mm								
12.5°	0	0	.....		.3 mm to .44 mm	.....	.44 mm	.....
43°	0	0	.....		.....	.....	.....	.....
54°	0	.54 to .85 mm	.....		.....	.....	.....	.....
60°	0	.....	Difference between fields = .51 mm		.....	.....	.....	.....

These results which agree in general are also shown in Fig. 13 for comparison with results obtained by other methods.

The comparative results of the skiascopic method are as follows:

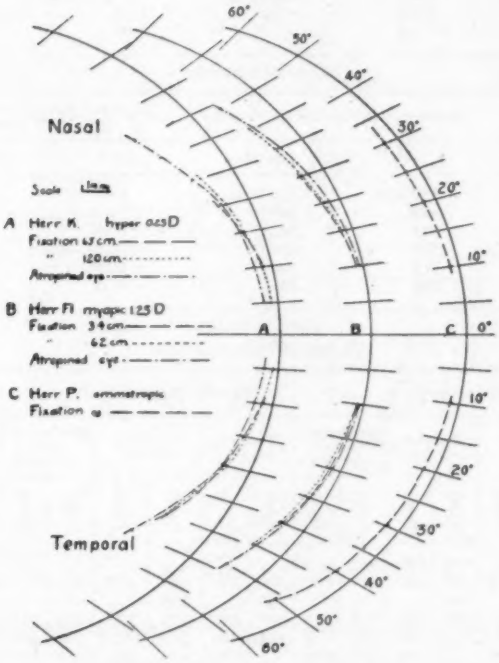
	Parent		Drualt			
			Nasal		Temporal	
Angle	Prim. field	Second. field	Prim. field	Second. field	Prim. field	Second. field
10°	.....	.....	−0.3 D	+0.1 D	−0.1 D	0.0 D
15°	−0.50 D	0.0 D	.....	.....	.....	.....
20°	−0.75	0.0	−1.4	+0.1	−0.4	+0.1
30°	−1.75	0.0	−2.9	+0.3	−1.2	+0.2
40°	.....	.....	−4.6	+1.0	−2.3	+0.8
45°	−2.75	+0.75	.....	.....	.....	.....
50°	.....	.....	−6.7	+2.1	−3.9	+2.0
Calculated Corresponding Primary and Secondary Image Distances from Retina in mm						
10°	.....	.....	−0.09	+0.03	−0.03	0.0
15°	−0.15	0.0				
20°	−0.22	0.0	−0.40	+0.03	−0.12	+0.03
30°	−0.46	0.0	−0.73	+0.07	−0.32	+0.045
40°	.....	.....	−1.0	+0.25	−0.53	+0.20
45°	−0.59	+0.17				
50°	.....	.....	−1.23	+0.49	−0.77	+0.46

These results agree very well except for the difference of the findings on the nasal side. This is probably due to the inclination of the visual to the optical axis as Drualt concludes.

These results are shown in graphic form in Fig. 13.

In considering the results of the subjective method, Peschel's eye was so myopic (6.6 D) that the position of his fields relative

FIG. 14.



Positions of the primary and secondary astigmatic image fields relative to retina as computed from Loria's data.

to his retina is of little value in comparison with the measurements on emetropic eyes. Wertheim got no results. In view of the results found by others and their substantial agreement, it must be concluded that he was in error.

The following tables give the results of Albini, Loria and Heinrich:

Albini's Results, Averaged.

Angle	Correcting lens		Calculated distance of secondary image from retina	
	Temporal	Nasal	Temporal	Nasal
18°	.....	+0.25 D	.....	+0.11 mm
21°	+0.37 D	+0.62	+0.31 mm	0.44
24°	+0.50	+0.75	0.15	0.45
27°	+0.87	+1.0	0.47	0.58
30°	+1.3	+1.4	0.85	0.93
33°	+1.6	+1.8	1.00	1.20
36°	+2.1	+2.4	1.37	1.50
39°	+2.7	+2.6	1.66	1.59
42°	+3.0	+3.0	1.75	1.77
45°	+3.5	.....	2.06	.....

Heinrich's Results.

Angle	Distance of vertical thread from eye		Calculated distance of primary image from retina	
	Nasal	Temporal	Nasal	Temporal
0°	350 mm	350 mm	0	0
1°	348	349	−0.02	−0.02
2°	341	343	−0.05	−0.05
3°	331	332	−0.09	−0.09
4°	315	320	−0.16	−0.14
5°	294	202	−0.24	−0.21

Loria's Results.

Object Distances in mm.

Angle	Herr K hypermetropic 0.25 D						Herr Pl. Myopic 1.25 D						Herr P emetropic	
	Fixation 65 cm.		Fixation 120 cm.		Eye treated with atropine		Fixation 34 cm.		Fixation 62 cm.		Eye treated with atropine		Fixation infinity	
	Nasal	Temp.	Nasal	Temp.	Nasal	Temp.	Nasal	Temp.	Nasal	Temp.	Nasal	Temp.	Nasal	Temp.
5°	...	...	625	650	675	675	...	...	....	....	...	...	...	...
10°	365	375	480	480	500	500	300	300	480	475	700	700	1000	1000
15°	250	...	325	...	...	...	250	...	385	...	550	...	...	...
20°	175	190	250	250	275	275	200	250	325	335	450	550	650	650
25°	...	152.5	175	160	175	175	175	225	237.5	287.5	350	400	600	...
30°	...	110	...	120	125	125	...	180	200	235	250	300	500	600
35°	...	...	...	...	100	100	...	...	170	182.5	200	225	...	...
40°	...	...	...	...	...	...	...	...	....	....	150	175	...	400
50°	...	...	...	...	...	...	...	...	....	....	...	...	...	200

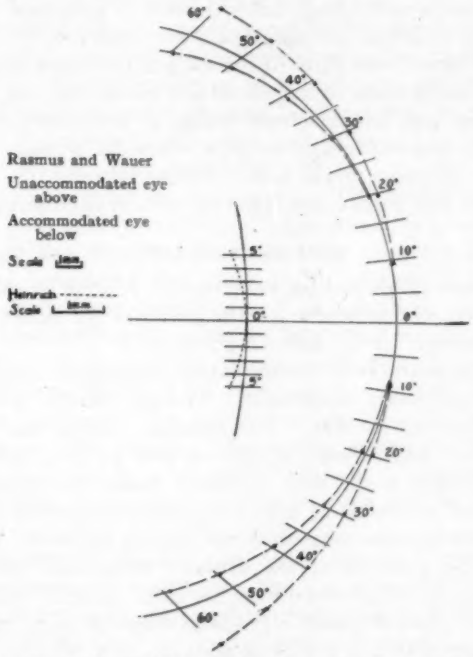
Calculated image distances in mm.

5°	...	...	0.23	0.21	0.50	0.50	...	...	....	....	...	...	...	...
10°	0.35	0.33	0.37	0.37	0.67	0.67	0.20	0.20	0.22	0.23	0.13	0.13	0.31	0.31
15°	0.71	...	0.65	...	...	...	0.42	...	0.40	....	0.28	...	...	...
20°	1.14	1.02	0.88	0.88	1.07	1.07	0.71	0.44	0.55	0.53	0.41	0.30	0.45	0.45
25°	...	1.30	1.27	1.40	1.53	1.53	0.91	0.57	0.86	0.67	0.60	0.50	0.47	...
30°	...	1.83	...	1.82	1.99	1.99	...	0.86	1.05	0.87	0.90	0.73	0.53	0.47
35°	...	...	...	...	2.30	2.30	...	...	1.24	1.15	1.12	0.99	...	...
40°	...	...	...	...	...	...	...	...	....	....	1.46	1.21	...	0.60
50°	...	...	...	...	...	...	...	...	....	....	...	...	...	1.02

The results are all shown graphically in Figs. 13, 14, and 15.

The theoretical work is too long and technical to attempt to summarize. The investigators all agree, however, that the retina lies between the primary and secondary astigmatic fields, approximately half way.

FIG. 15.



Positions of the astigmatic image fields as computed from data given by Rasmus, Wauer and Heinrich.

Fig. 15 shows Rasmus' and Wauer's calculated position of the primary and secondary fields relative to the retina for the accommodated and unaccommodated eye. Their findings are in agreement with those of Mattheissen.

CONCLUSION.

By comparison it will be seen that the results obtained by the different experimental methods in general agree. The separation

of the primary and secondary astigmatic image fields as found experimentally is somewhat less than that found by calculation.

The conclusion from all work to date is:

First, that the image of oblique rays formed by the lens system of the eye is astigmatic;

Second, that the retina lies between the primary and secondary astigmatic image fields formed by a surface in space, all parts of which are equidistant from the eye;

Third, that those parts of the astigmatic image bundle which are in the form of tangential and radial lines are sufficiently sharp and the peripheral retina is sufficiently sensitive to make it possible to determine when the primary or secondary astigmatic focus falls upon the retina;

Fourth, that monocular space perception is tridimensional.

PURPOSE OF RESEARCH.

The facts which had been determined relative to oblique astigmatism were not sufficient for the purposes for which this work was undertaken for the following reasons:

1st. The position of the secondary astigmatic image field had not been sufficiently determined. Peschel's results were vitiated by his abnormal myopia. The ophthalmoscopic and skiascopic results were not sufficiently definite and no measurements had been made close to the axis. Albini's results are questionable in that he used a number of dots for a stimulus instead of a radial line, nor did he make measurements close to the axis.

2nd. The position of the primary astigmatic field had not been sufficiently determined near the axis. Loria's readings only go in to 10°, and Heinrich's readings from 0° to 5° are by only one observer who had 2.5 D of myopia.

3rd. No determination has been made of the position of the astigmatic fields for light of different wave-length. The paraxial stimuli used have all been illuminated by white light in both the objective and subjective method. This excepts Albini's attempts, which were made valueless, as he used disks of colored paper which were round in shape.

METHOD OF MAKING MEASUREMENTS.

In designing the apparatus it was thought that it might be possible to determine the inclination of rays which constitute the

primary and secondary astigmatic image focus by means of three monochromatic rays of small cross section whose inclination could be controlled as described on page 30.

One of the rays would strike the eye at any desired obliquity and pass through its axis. A second ray whose inclination and separation from ray (1) could be controlled would strike the eye in say a horizontal plane to the left or right of ray (1). A third ray whose inclination could be controlled independently from rays (1) and (2) would strike the eye in a vertical plane above or below ray (1). The position of the primary astigmatic focus for a given obliquity and zone could be determined by the inclination of ray (2) when the image of rays (1) and (2) were superimposed. In the same way the position of the secondary astigmatic focus could be determined. The apparatus was adjusted for such measurements and a few attempts to use it made. The method was discarded because of the apparent difficulty of setting and because of the great length of time it would take to get the desired measurements by such a method. It has, however, not been sufficiently tested to prove that it cannot be used successfully. It was decided to use in its place the method by which Loria and Heinrich got such definite results. That is, by line sources the rays from which could be caused to fall on the eye at any desired obliquity; the reading where the line focuses most sharply when in a tangential position giving the primary astigmatic field; that where it focuses when in a radial position giving the secondary astigmatic field.

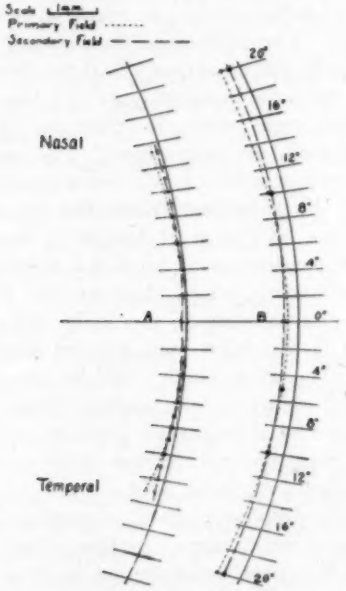
The apparently simplest method of getting a tangential and radial source was to substitute a fine line, which could be placed in a vertical and horizontal position, for the hole in Mirror (E), Fig. 1. This, however, could not be used because due to the 45° inclination of Mirror (E) the two ends of the line source when in a horizontal position would be in different focus.

A tangential and radial line source was first obtained as follows: With a point source in (E), Fig. 1, very narrow horizontal and vertical slits were placed at (H), Fig. 1. The effect of a very narrow horizontal slit was to spread the point source into a vertical line by diffraction effect. The apparent position of this line in space is controlled by moving Mirror (F), Fig. 1. Similarly a very narrow vertical slit gives a horizontal line.

The method of making observations was first to properly posi-

tion and accommodate the eye. With the rotating system set to give zero obliquity, readings were made. The system was then rotated to give any desired obliquity and locked, and another set of readings taken. In making the zero readings, allowance has to be made for the lateral displacement of the beam to the right by the plane parallel (Q), Fig. 1. To do this the displacement

FIG. 16.



(*A*) shows positions of primary and secondary astigmatic image fields relative to retina from measurements of Mr. Ames' eye using a red (λ6560) point source and the diffraction effect of very narrow horizontal and vertical slits to obtain vertical and horizontal line stimuli.
(*B*) shows positions of primary and secondary astigmatic image fields relative to retina from measurements of Mr. Ames' eye using a green (λ5461) source and the improved apparatus.

of the beam was first calculated. Another small white speck is placed on the front of (Q), the calculated distance to the left of the speck already there. In making zero obliquity readings the fixation line is centred in the diffusion disk from the right-hand speck. In making all other readings it is centred on the diffusion disk from the left-hand speck.

A set of readings, see Table VII, was taken in September,

1919, by this method. The hole in (E) was illuminated by red (λ 6563) light. The readings were obtained from the settings of (F), Fig. 1, where the horizontal and vertical lines appeared to be most sharp. Both the nasal and temporal side of the lens were measured at the angular obliquity shown.

TABLE VII.

Oblique Astigmatism. (*Mr. Ames, Red Stimulus.*)

Observer Mr. Ames, right eye, accommodation relaxed. Color of paraxial stimuli red (λ6563). Focal length of mirror = 71.9 mm.

Angular obliquity of source	Primary astigmatic field			Secondary astigmatic field		
	Average readings (distance from focus of mirror), mm	Calculated object distance, mm	Calculated image distance from retina, mm	Average readings (distance from focus of mirror), mm	Calculated object distance, mm	Calculated image distance from retina, mm
Temporal						
0°	+0.32	−1600	−0.018	−1.99	+2770	0.104
2°	−0.18	∞	0.000	−1.06	+5180	0.057
3°	−0.55	+9600	+0.030	−1.27	+4350	0.066
4°	−0.58	9100	0.032	−1.57	+3530	0.082
5°	−0.66	8060	0.036	−1.83	+3000	0.096
6°	−1.06	5200	0.055	−1.44	+3840	0.075
7°	−1.62	3400	0.082	−1.57	+3530	0.082
8°	−2.38	2350	0.123	−1.49	+3700	0.078
9°	−2.99	1940	0.150	−1.75	+3140	0.092
10°	−3.96	1510	0.176	−2.55	+2210	0.129
11°	−4.53	1345	0.213	−2.37	+2370	0.122
12°	−6.26	1035	0.277	−2.82	+2030	0.142
13°	−7.00	946	0.302	−3.08	+1890	0.152
Nasal						
0°	+0.24	−22500	−0.014	−1.99	+2770	0.104
2°	+0.58	− 7400	−0.039	−1.82	+3030	0.095
3°	−0.40	+13300	+0.022	−1.33	+4170	0.069
4°	−0.77	7000	0.041	−1.36	+4070	0.071
5°	−1.08	5080	0.059	−1.40	+3950	0.073
6°	−2.02	2650	0.110	−1.43	+3870	0.075
7°	−1.51	3650	0.081	−1.49	+3700	0.078
8°	−2.21	2500	0.115	−1.53	+3600	0.080
9°	−2.34	2380	0.121	−0.99	+5540	0.052
10°	−2.56	2200	0.131	−1.00	+5480	0.053
11°	−3.34	1760	0.163	−1.07	+5140	0.056
12°	−4.49	1380	0.211	−1.33	+4170	0.069
13°	−4.05	1480	0.194	−1.16	+4750	0.061

Fig. 16 (A) shows the position of the primary and secondary image fields as determined relative to the retina.

Analysis of these findings will be deferred until later. Great care must be taken to hold the accommodation constant and practice is required in making the observations. Loria also found this necessary, see (23).

It was thought desirable to make similar readings, using actual line sources. Accordingly, a lens of 125.6 mm focal length was substituted in the place of mirror (F), Fig. 1, and a slit .03 to .05 mm wide and 4.0 mm long was placed behind it. The slit was arranged so that it could be placed in either a horizontal or vertical position and illuminated by monochromatic light. The method of making the measurements was the same as before. The measurements which were made with light of four different wave-lengths are given in Table VIII and Fig. 17.

It was thought that there might be a tendency of the eye to accommodate for these peripheral stimuli (see Heinrich (21)); that the crystalline lens might flatten to bring the first astigmatic focus on the retina and become more convex to see the second focus more clearly. It was known that this could not be of large amount as the monochromatic central fixation cross remained

TABLE VIII.

Oblique Astigmatism. (*Mr. Ames, Stimuli of Various Wave-lengths.*)
Observer Mr. Ames, right eye, accommodation relaxed. Focal length of lens = 125.6 mm.

(Primary Astigmatic Fields.)

Angular obliquity of source	Red (λ6563)			Yellow (λ5893)			Green (λ5461)			Blue (λ4861)		
	Average readings, mm	Object distance, mm	Image distance, mm	Average readings, mm	Object distance, mm	Image distance, mm	Average readings, mm	Object distance, mm	Image distance, mm	Average readings, mm	Object distance, mm	Image distance, mm
Temporal												
0°	−0.045	286000	0.001	−5.70	3000	0.095	−8.93	2000	0.144	−18.40	1100	0.260
2°	1.39	11500	0.026	.....	....	....	10.11	1800	0.161	21.45	980	0.291
3°	.....	.....	....	6.03	2840	0.101	....	....	....	....	....	....
4°	1.60	10100	0.029	6.90	2520	0.115	10.62	1730	0.167	23.15	928	0.308
6°	5.97	2880	0.101	8.45	2090	0.138	12.04	1550	0.185	25.10	877	0.325
8°	8.98	1990	0.144	11.76	1570	0.184	16.40	1200	0.239	28.55	805	0.353
10°	11.6	1660	0.173	14.75	1310	0.221	21.52	978	0.292	33.20	732	0.388
12°	15.4	1260	0.226	17.31	1150	0.249	27.62	824	0.346	41.50	669	0.426
14°												
16°		Readings prevented by blind spot.										
18°												
20°	21.8	967	0.294	22.75	939	0.304	40.60	656	0.432	46.90	644	0.430
Nasal												
0°	−0.045	286000	0.001	−5.70	3000	0.095	−8.93	2000	0.144	−18.40	1100	0.260
2°	0.92	17100	0.017	.....	....	....	.....	....	....	19.7	1040	0.274
3°	.....	.....	....	6.35	2720	0.107	10.26	1780	0.162	....	....	....
4°	2.93	5590	0.052	7.23	2420	0.119	10.61	1730	0.167	20.7	1010	0.285
6°	3.76	4410	0.066	9.03	1970	0.147	13.40	1420	0.202	23.6	914	0.302
8°	4.65	3610	0.081	9.48	1890	0.153	14.54	1320	0.216	25.0	880	0.324
10°	7.00	2490	0.115	11.01	1660	0.173	15.87	1230	0.233	25.3	872	0.327
12°	8.36	2110	0.135	12.11	1530	0.187	17.17	1160	0.247	29.3	792	0.359
14°	11.52	1590	0.182	15.02	1290	0.224	21.71	964	0.296	30.9	765	0.372
16°	13.13	1430	0.201	17.95	1120	0.255	25.65	864	0.330	37.9	678	0.428
18°	16.05	1220	0.235	17.19	1160	0.247	29.43	792	0.359	38.3	674	0.430
20°	18.10	1110	0.258	24.63	888	0.322	28.95	800	0.356	45.0	652	0.434

TABLE VIII—(*Continued*).
(Secondary Astigmatic Fields.)

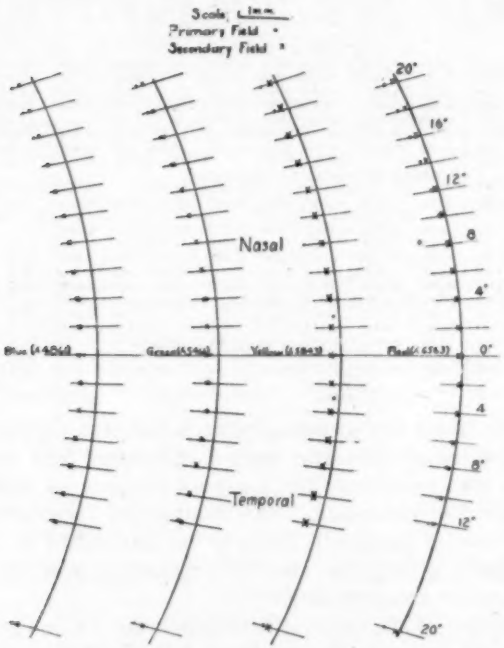
Angular obliquity of source	Red (λ6563)			Yellow (λ5893)			Green (λ5461)			Blue (λ4961)		
	Average readings, mm	Object distance, mm	Image distance, mm	Average readings, mm	Object distance, mm	Image distance, mm	Average readings, mm	Object distance, mm	Image distance, mm	Average readings, mm	Object distance, mm	Image distance, mm
Temporal												
0°	−4.11	4030	0.072	−10.93	1670	0.172	−15.12	1310	0.218	−27.10	834	0.342
2°	2.52	6450	0.045	9.64	1860	0.161	12.99	1440	0.199	23.95	906	0.315
4°	2.42	6710	0.043	9.86	1830	0.158	13.97	1360	0.210	23.80	909	0.314
6°	3.47	4760	0.061	9.78	1840	0.156	14.91	1290	0.221	23.52	917	0.311
8°	5.11	3300	0.087	11 54	1590	0.180	17.11	1160	0.247	25.25	875	0.326
10°	6.17	2790	0.104	14.17	1350	0.213	19.67	1040	0.274	31.92	750	0.379
12°	7.50	2340	0.123	17.01	1170	0.246	21.84	967	0.298	31.75	753	0.378
14°												
16°	Readings prevented by blind spot.											
18°												
20°	3.97	4180	0.069	13.61	1390	0.206	19.57	1050	0.272	27.05	835	0.341
Nasal												
0°	4.11	4030	0.072	−10.93	1670	0.172	−15.12	1310	0.218	−27.1	834	0.342
2°	2.94	5590	0.052	9.86	1830	0.158	13.38	1410	0.203	25.9	860	0.331
4°	2.84	5750	0.050	10.03	1800	0.160	14.20	1340	0.214	25.1	877	0.325
6°	3.64	4550	0.064	9.73	1850	0.156	14.01	1360	0.211	26.5	846	0.337
8°	3.30	4980	0.058	10.15	1780	0.162	13.92	1370	0.210	25.2	875	0.324
10°	3.38	4890	0.059	11.61	1580	0.181	15.10	1280	0.224	24.9	881	0.321
12°	4.27	3920	0.074	11.90	1550	0.185	15.33	1260	0.227	26.9	838	0.340
14°	4.42	3790	0.076	12.53	1490	0.193	17.39	1150	0.250	27.3	830	0.343
16°	5.37	3150	0.093	12.52	1490	0.193	17.94	1120	0.255	24.5	890	0.320
18°	5.74	2990	0.097	8.15	2160	0.133	16.67	1180	0.242	22.9	935	0.305
20°	4.28	3890	0.074	7.39	2370	0.122	14.06	1360	0.212	21.1	992	0.288

sharp. To avoid any such possibility, however, an apparatus was constructed which presented both a tangential and radial line source at the same time, the apparent distance of each source being controlled separately. This enabled the focus of both the first and second astigmatic fields to be determined at the same time, thereby giving the absolute separation between the two independent of accommodation.

A drawing of the optics of this apparatus is given in Fig. 18. Units A and B are made up of two 6 V lights (a′) and (b′), which illuminate plates (a) and (b), back of which are placed monochromatic filters. Plates (a) and (b) are silvered and on plate (a) a series of vertical lines is scratched, and on plate (b) a series of horizontal lines. The units A and B can be moved forward and back independently of one another. The light from A and B is made parallel by the two lenses L and L and after passing through the prism system C, with its half-silvered surface (c) follow the same path HE. The eye is placed at (E), about which centres the units AB and C turn. The obliquity at which

the ray HE strikes the eye is given by the position of the pointer P on the scale SS. D is a plane parallel which reflects into the eye the monochromatic fixation cross from the system S, which is similar to the system described in the other apparatus. F is a very small glass bead illuminated by a light placed at K. It

FIG. 17.



Positions of the primary and secondary astigmatic image fields relative to the retina from measurements on Mr. Ames' eye using vertical and horizontal line stimuli of the wavelength given.

serves to centre the eye in the same manner as the speck on the plane parallel described in the other apparatus. DFK and system S remain stationary when the units A, B and C are turned about the centre E. To keep the eye fixed in a forward and back position, a mouth grip and forehead rest were used instead of the method described on page 29. The mouthpiece was an indentation of the observer's teeth made in sealing wax on a brass plate. This

and the forehead rest were so positioned that the eye came in the proper place. This is a possible arrangement where the beam used is larger than the diameter of the pupil. It has the advantage of being somewhat simpler but is not as exact as the other method.

With this apparatus both vertical and horizontal lines of the same obliquity are made to fall upon the eye at the same time. As the apparent position of each can be controlled independently,

FIG. 18.

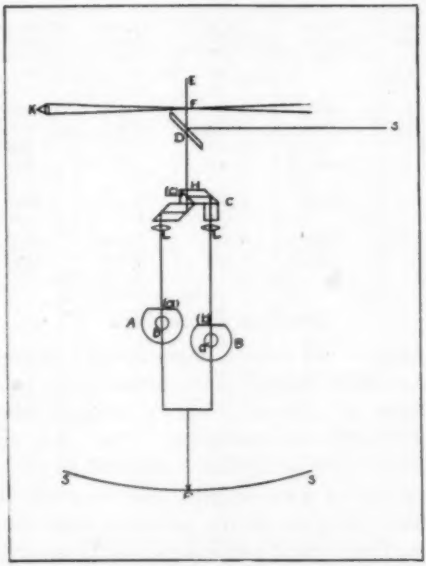


Diagram of improved apparatus for measuring oblique or field astigmatism.

it is possible to get a sharp image on the retina of both sets of lines at once.

Up to date only a set of test readings has been made with this apparatus, see Table IX and Fig. 16B. These readings were made to see how measurements with this method checked up with former measurements. Readings were taken in which the vertical and horizontal lines were used separately, that is, in the same method which was followed with the old apparatus, and also

where the vertical and horizontal lines were both judged to be sharp at the same time. No difference was found.

It will be seen by comparing the curves in the figures that the results obtained by both methods are substantially the same.

TABLE IX.

Oblique Astigmatism. (Mr. Ames, with Improved Apparatus.)

Observer Mr. Ames, right eye, accommodation relaxed. Color of paraxial stimuli Green (λ5461).

Angular obliquity of source	Primary astigmatic field			Secondary astigmatic field		
	Average readings, mm	Object distance, mm	Image distance, mm	Average readings, mm	Object distance, mm	Image distance, mm
Temporal						
0°	−11.9	1326	0.216	−14.7	1070	0.269
5°	13.3	1190	0.229	15.4	1026	0.252
10°	20.2	791	0.360	17.7	890	0.320
20°	26.1	606	0.476	18.6	847	0.336
Nasal						
0°	11.9	1326	0.216	14.7	1070	0.269
5°						
10°	17.8	895	0.319	14.25	1102	0.259
20°	20.8	758	0.375	11.6	1348	0.213

ANALYSIS OF RESULTS.

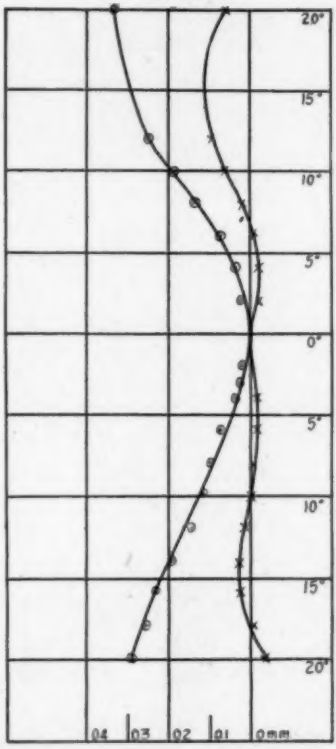
The similarity of the curves showing the position of the two astigmatic image fields relative to the retina found by the different methods, see Figs. 16 and 17, is very conclusive proof that they are substantially correct measurement of the conditions that exist in Mr. Ames' eye. This is further confirmed by the similarity of the curves for different wave-lengths, and the conformity of these curves with the difference in the spherical aberration measurements in the horizontal and vertical meridian which is explained in the next paragraph.

At first sight one is struck by two features of these curves:

First, that the two fields do not meet at the axis. On consideration, however, it will be seen that this was to be expected in view of the corneal astigmatism in Mr. Ames' eye, which was determined very accurately and is shown in the difference of the measurements for spherical aberration on the horizontal and vertical meridians. If allowance for this separation is made and the two fields for one wave-length moved relatively to each other until they coincide at the axis, they will take the position shown

in Fig. 19. In this figure the scale of the image distance has been increased relative to that of the angular obliquity to show more clearly the characteristic form of the curves, especially that of the secondary field which was found in all measurements. The

FIG. 19.



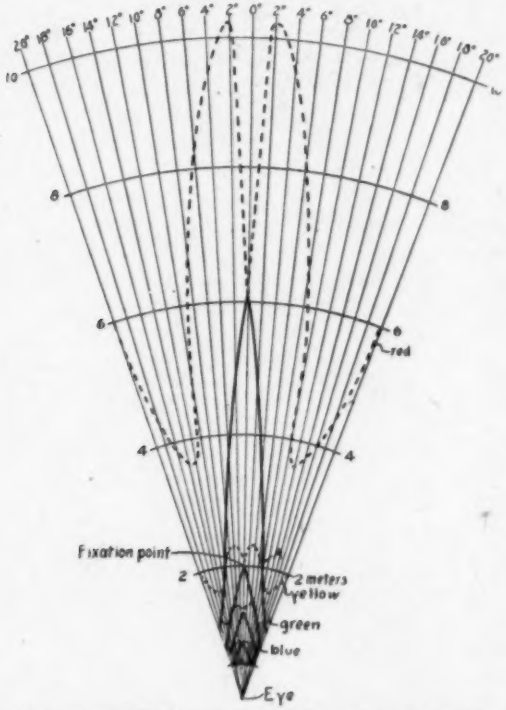
Curves showing the position of the primary and secondary astigmatic fields relative to the retina from the combined results of Mr. Ames' readings for all wave-lengths, correction having been made for chromatic aberration. Correction has also been made for Mr. Ames' corneal astigmatism so that the two fields coincide with the retina on the axis.

two fields were brought together at the axis and show the position of the fields in an eye free from corneal astigmatism.

Second, that the fields on the nasal and temporal side do not lie in the same position relative to the retina. This is due either to the difference between the visual and optical axis (angle alpha)

in Mr. Ames' eye or to a lopsideness in the shape of his lens or retina or both. The angle alpha in Mr. Ames' eye amounts to 3° on the horizontal meridian, and the amount of asymmetry of the astigmatic curves suggests that it may be due to the tipping of the lens system relative to the bulbus of the eye. The relation-

FIG. 20.



Curves showing the position in space of the primary and secondary astigmatic object fields; that is where tangential and radial lines of the colors given will focus on the retina when the eye is focused for yellow at a point two meters distant. These curves are constructed from the averages of all Mr. Ames' readings, corrections being made for his corneal astigmatism so that the primary and secondary image fields will coincide on the axis.

ship between the angle alpha and the position of the astigmatic fields would be an interesting matter to determine. It would require, however, the readings from a number of eyes.

So far we have been considering only the position of the primary and secondary astigmatic image fields. Corresponding

to these different colored image fields there are object fields in space. They are shown in Fig. 20. The position of the primary astigmatic object field for one color is such that with any given accommodation of the eye all point sources of the same color in that field will be imaged as tangential lines on the retina. The position of the secondary astigmatic object field is such that with a given accommodation all point sources in that field will be imaged as radial lines on the retina. The position of these fields for light of different wave-length is different. See Fig. 20 where the position of these object fields for red, yellow, green and blue is given. With a change of accommodation the position of all these fields shifts.

Consider the nature of the image of a white light point source situated at A, Fig. 20. Being on the first astigmatic object field for red, the red light in it will be stretched in a tangential direction. Being in the second astigmatic image field for yellow, the yellow light will be stretched in a radial direction. Being beyond the second astigmatic field for green and blue, the green and blue light in it will be in the form of a radial diffusion circle. The images formed by light of intermediate wave-lengths will take intermediate shapes.

Images of a similar nature will be formed by points lying in a circle formed by revolving point A about the visual axis. With the same accommodation, however, there is no other position in space, either nearer or farther or at a lesser or greater angle of obliquity where a point source will form a similar image. The image of a point source in any other position in space, though different, will have its characteristic form. It follows that:

1st. With any given accommodation every object in space has its characteristic retinal image form due to its position in space relative to the fixation point.

2nd. This characteristic image form is governed both by the shape and color of the object.

A very definite basis is thus given for monocular tridimensional space perception. Heinrich (24), page 60, was the first to recognize this possibility, but had not carried his experimentation far enough to discover the effect of the secondary astigmatic fields and that of color.

A photographic lens with the same achromatism and oblique astigmatism as have been found in Mr. Ames' eye is being con-

structed. It is hoped that with this the subjective nature of monocular depth perception as based on the above described characteristic imaging can be more definitely ascertained.

CONCLUSION.

It is recognized that this work on oblique astigmatism, as is the case with that on spherical and chromatic aberration, is hardly more than preliminary.

Measurements should be made on a large number of eyes both normal and abnormal. These should be made for various accommodations and with various sizes of pupil. The relationship between angle alpha and the position of the astigmatic fields relative to the retina should be determined.

There is another very important collateral research that should be undertaken in view of the above findings. That is, the determination of the color sensitivity of the peripheral retina.

In the voluminous research that has been done on this subject great stress has been laid upon the necessity of equating the intensity of the different colors used for objective stimuli. No attention (with one exception, Albini (17)) has been given the nature of the image that falls on the peripheral retina. To make any comparison of the relative sensitivity of the peripheral retina to different colors it is necessary that the images of the different colors shall have the same intensity.

As far as is known in making measurements of peripheral color sensitivity, 1st, the accommodation of the observer has always been kept constant; 2nd, the distance of the different color stimuli from the observer has always been kept constant (with the exception of Albini's work); 3rd, the position of the fixation or accommodation object and the distance of the peripheral color stimuli was the same. This distance for the sake of convenience was not great, usually less than one metre.

From Fig. 20 it can be seen that with a given accommodation the nature of the images on the retina from stimuli of different colors all the same distance away would be very different both in shape and in intensity.

If, for example, a stimulus 1 cm in diameter and one metre distance were used, and the eye were accommodated for red, the mean intensity of the image on the fovea of a red stimulus would be approximately twice that of a blue. If the eye were

accommodated for blue, this difference would be reversed. Similar differences in general greater in amount would exist in the case of peripheral stimuli.

In view of these facts all past measurements on peripheral color sensitivity may be of no value. At least their validity is put in question until measurements are made in which the intensity of the retinal image formed by the different colored stimuli are equated. It may be of interest that the late Dr. J. W. Baird, an authority on the subject, shared this view.

The simplest way to do this would be with an apparatus as described on page 24 and following. The primary or secondary astigmatic object field of the observer would be first determined for the different colors; the brightness of the different color sources equated; then either a radial or tangential line source could be used, its apparent position in space being so controlled by a lens that the first or second astigmatic image for the different obliquities would fall on the retina. A change in the length and width of the line source would have to be made for the different colors to compensate for the change in size of image in its different positions in space.

DISTORTION.

Distortion is that characteristic of a lens which causes variation in distance between points in the image plane which in the object plane are equidistant. The human eye is subject to so-called barrel distortion, which means that in the image plane distances between points which are equidistant in the object plane keep decreasing from the centre outward.

The existence of barrel distortion in the eye has been recognized for a long time. The work that has already been done both theoretically and experimentally is of such a nature that it was believed that it could not be improved upon. It was therefore decided to carefully tabulate the results and use the findings.

The articles considered will be taken up chronologically.

(1) Helmholz [5] described a series of experiments which show how this distortion can be seen. It is apparent in causing a straight line at one side of the axis of vision to bow out in its central portions. A strip 8 to 10 centimeters wide when its centre is fixated will appear wider in its middle than at its ends. He

[5] "Handbuch der Phys. Opt."

also constructed a hyperbolic chess board on a theoretical basis, which when held twenty centimeters from the eye assumes the appearance of straight lines when fixated in the centre.

(2) Matthiessen,[6] in his profound article "Uber die geometrische Gestalt des theoretischen Retina der periskopschen Schematischen Auges," gives theoretical data from which the distortion of the eye can be theoretically calculated. The corresponding external and internal angles have been calculated and are as follows:

External angle	Internal angle	External angle	Internal angle
0°	0°	50.32°	40.16°
3.20°	2.40°	53.25°	42.26°
6.40°	5.21°	56.15°	44.35°
9.58°	8.02°	59.05°	46.42°
13.16°	10.44°	61.54°	48.47°
16.33°	13.24°	64.37°	50.47°
19.53°	16.04°	67.17°	52.45°
23.09°	18.41°	69.55°	54.38°
26.21°	21.15°	72.29°	56.30°
29.29°	23.46°	75.00°	58.19°
32.35°	26.14°	77.29°	60.04°
35.38°	28.40°	79.56°	61.47°
38.40°	31.03°	82.22°	63.28°
41.41°	33.25°	84.46°	65.07°
44.41°	35.44°	87.12°	66.45°
47.39°	38.02°		

Curve (A), Fig. 21, shows this distortion in graphic form.

(3) Donders[7] made measurements on a patient affected with exophthalmia, *i.e.*, protruding of the eyeball. He projected a light into the eye at a known angle and measured the distance from the corneal margin to the point on the schlera where the image was formed. The following is a set of measurements he made on an eye which, except for the exophthalmia, was normal.

Angular obliquity of light	Distance of image from the corneal margin	
	Temporal	Nasal
50°	15.3 mm	15.7 mm
60°	13.2	13.5
70°	11.2	11.6
80°	9.3	9.5
90°	8.0	8.0

[6] Arch. für Oph. 25, p. 257 (1879).

[7] "Die Grenzen des Gesichtsfeldes in Beziehung zu denen der Netzhaut," Arch. für Ophth. 23, p. 255 (1877).

(4) Drualt[8] reviews the work of Groenouw and Donders. He finds their measurements substantially similar to his and to those found theoretically. They all show that for paraxial objects the posterior nodal point of the eye changes.

He gives theoretical determinations and measurements on dead and living eyes. His methods of measurement are the same as Donders.

In his calculations he uses an eye having the form given by Merkel with refractive surfaces having the shape and position given by Tscherning.[9] Diameter of bulbus 24.4 mm, that of corneal, 12 mm.

Angular obliquity of light	Distance of image from corneal margin
20°	20.4 mm
40°	17.2
60°	13.4
80°	9.7
100°	7.3

His measurements on dead eyes which he said were "very fresh" were as follows:

Angular obliquity of light		Distance of image from corneal margin	
		Temporal	Nasal
I.	30°	18.2 mm.	....
	40°	16.8	....
	50°	15.2	14.5 mm.
	60°	13.5	13.0
	70°	11.4	11.4
	80°	9.7	10.
	90°	8.4	8.8
II.	30°	19.6	....
	40°	17.6	18.3
	50°	15.8	16.4
	60°	13.0	14.4
	70°	10.6	12.0

The diameter of the bulbus in Case I was 23 mm, that of cornea 10.3 mm; in Case II, 25 mm and 11 mm.

[8] "Note sur la Situation des Image retiennes formées par les Rayons tres obliques sur l'Axe optique," Arch. d'Ophth., 18, p. 685 (1898).

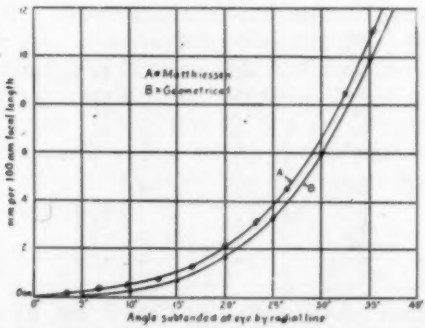
[9] Oeuvres de Young, 1896, p. 134.

His measurements on live eyes were on two patients affected with exophthalmia. He measured both eyes of each, and as they were all approximately the same, he gives the average results of the two eyes. They are as follows:

Angular obliquity of light	Distance of image from corneal margin	
	Temporal	Nasal
III. 50°	15.2 mm.	....
60°	13.5	....
65°	.....	14.0 mm.
70°	11.6	13.0
80°	9.7	11.1
85°	9.2	....
90°	.....	9.7
100°	.....	8.7
IV. 50°	16.0	....
60°	14.1	....
70°	11.8	14.2
80°	10.0	12.7
90°	8.5	11.2
100°	.....	9.9
105°	.....	9.3

The diameter of the bulbus in Case III was 26.5 mm, the horizontal diameter of the cornea 11.7 mm.

FIG. 21.



Distortion, curves show apparent shortening of radial line from axis to designated angle. Curve *B* shows distortion due to spherical shape of retina. Curve *A* shows distortion due to both spherical shape of retina and optical effect of lens system as calculated by Matthiessen.

Drualt made measurements on two other people and found similar results.

(5) Tscherning [10] describes optical distortion and gives an account of his experiments with an artificial eye, in which he finds a distortion similar to that described by Helmholtz.

SUMMARY.

The results of Donders and Drualt on the experimental side are substantially similar to those calculated by Matthiessen. The difference on the temporal and nasal side is probably due to the existence of the angle alpha in the eyes measured. If we take the average of all the readings we get the following table:

Angular obliquity of light	Distance of image from corneal margin	Corresponding internal angle
30°	18.9 mm	24.5°
40°	17.6	32.3°
50°	15.5	39.4°
60°	13.5	47.8°
70°	11.8	54.4°
80°	10.2	61.0°
90°	8.9	67.0°

The internal angles which correspond with the above distances of images from the corneal margin have been calculated as accurately as possible and are given in the right-hand column. It will be noted by comparing the external and internal angles with those of Matthiessen on page 80 that their results agree very closely.

The computed results of Drualt are very similar. (See tables, page 81.)

It may be of interest to consider the cause of this distortion. Curve (B), Fig. 21, shows the distortion which would exist in the human eye due solely to the effect of the spherical shape of the retina. This might be called geometrical distortion. By comparison with curve (A) it is seen that this geometrical distortion alone is not as great as that which exists in the eye. It can be concluded therefore that the distortion caused by the optical system is also barrel in its nature and increases that caused by the spherical shape of the retina.

Fig. 22 shows the distortion of the eye as found in another form. The barrel-shape grid shows the appearance to the eye

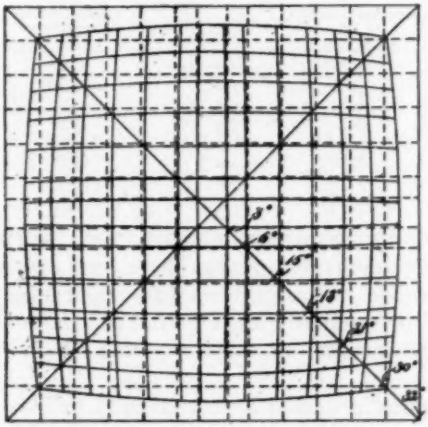
[10] "Physiologic Optics," published by The Keystone Press, 1904. Chapter xiv.

which a rectilinear grid would have if its center was fixated and if it was viewed at such a distance that the points on the lower right-hand diagonal subtended angles to the visual axis as indicated.

FINAL CONCLUSION.

As was stated in the preface the researches described were undertaken for the purpose of determining the exact nature of the image received by the human eye in the belief that a knowledge of its nature would be of aid in suggesting how the various parts

FIG. 22.



Grids showing distortion in eye. The barrel-shaped grid shows the amount of distortion the rectilinear grid suffers when viewed by the eye.

of a picture should be painted to give a technically pleasing and artistic effect.

As has been shown, the most important characteristics have been approximately determined, *i.e.*, spherical aberration, chromatic aberration, oblique astigmatism and distortion. A photographic lens embodying these characteristics is being constructed. It is believed that in connection with the three-color process a substantial reproduction of the retinal picture can be obtained.

WILDER LABORATORY,
DARTMOUTH COLLEGE,
HANOVER, N. H.

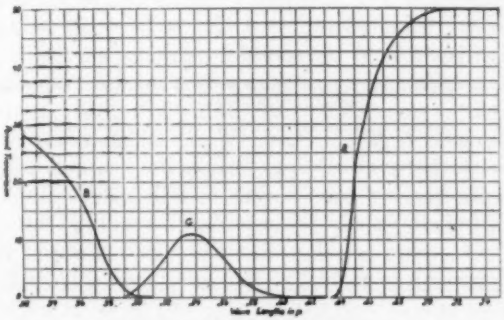
A COMPARISON OF MONOCHROMATIC SCREENS FOR OPTICAL PYROMETRY.

BY

W. E. FORSYTHE.

In working with any optical pyrometer it is generally sufficient to use a so-called monochromatic screen between the eye and the pyrometer filament or the comparison source in order that brightness comparisons can be made without trouble due to color differences. For the most part red glass screens have been used in the past, and the question is often asked—Why not use screens of other colors, as, for instance, a green or a blue screen? In connection with some other work quite a number of readings have been made with particular red, green and blue screens before the eyepiece of a disappearing-filament optical pyrometer. Since the above readings with the different glasses extended over

Fig. 1.



Spectral transmission of different screens.
Curve *B*, two thicknesses blue uviol glass, total thickness 3.9 mm.
Curve *G*, two thicknesses green glass, total thickness 5.2 mm.
Curve *R*, two thicknesses Jena red glass, total thickness 6.8 mm.

a wide range of temperatures, it was thought worth while to make a study of these readings to see if settings made with any one of these glasses as the monochromatic screen were more accurate than with either of the others.

The spectral transmissions of the three glass screens used are shown in Fig. 1. The red glass consisted of two thicknesses of

Jena Glass (total thickness 6.8 mm). This is the glass that has been used in Nela Research Laboratory as a standard red screen for pyrometric work for the past six or seven years. The green glass consists of two thicknesses (total thickness 5.2 mm) of an unsaturated green glass that was obtained from Mr. Gage of the Corning Glass Works. The blue glass consisted of two pieces of blue uviol glass (total thickness 3.9 mm).

In Table I is given the percentage of the total light from a black body at different temperatures that is transmitted by each of these glasses. These values were obtained by calculation, using the following formula:

$$T_R = \frac{\int_0^\infty J(\lambda T)\, V_\lambda\, T' d\lambda}{\int_0^\infty J(\lambda T)\, V_\lambda\, d\lambda}$$

where $J(\lambda T)\, d\lambda$ equals black-body energy as given by Wien's equation for the interval λ to $\lambda + d\lambda$, V_λ = visibility and T' the spectral transmission of the glass used. This integral can be calculated by the step-by-step method with sufficient accuracy for this purpose.

TABLE I.

Percentage of the Total Light from a Black Body at Different Temperatures Transmitted by Different Glasses.

Temperature of black body	Kind of glass		
	Jena red two thicknesses	Green two thicknesses	Blue uviol two thicknesses
	per cent.	*per cent.*	*per cent.*
1000°K	6.6	1.4	0.004
1800	2.1	3.2	.06
3000	1.0	4.2	.2

The values given in Table I show very well one reason for the preference for the red glass at low temperature. It will be noticed that at 1000° K, which is about as low as readings are generally made with the optical pyrometer, the red glass transmits much more of the light than does the green or blue glass. The table also shows why it is practically impossible to make readings through the two blue glasses at the very low temperature. At high temperatures the green glass transmits much more than does the red glass, but even the red glass as used on the pyrometer for

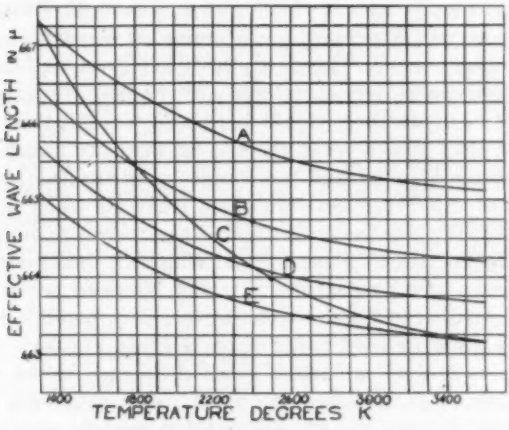
temperature ranges, roughly 1500° to 1800° K, transmits enough light to very greatly tire the eye.

If two red glasses such as described are being used in the eyepiece to observe a source at a temperature of 2000° K and one is removed, about twice as much light will reach the eye as before. If for the same condition, the two green glasses are being used and one of them removed, about 4.3 times as much light will reach the eye as before; while about 5.5 times as much light is transmitted by one thickness of blue glass as is transmitted by two thicknesses. These ratios do not vary but a small amount for different temperatures of the source observed. Corning high transmission red (50 per cent.) is a somewhat better glass than the Jena Glass here used. There may be better green or blue glasses than the ones used here, but these are the best we have found. With any one of the three glasses here used, a single thickness will not give a color match when there is an intensity match between the pyrometer filament and the source being studied for a large temperature difference between them, that is, when a rotating sector of small transmission is used. It is, therefore, necessary to use two thicknesses of glass. Even with two thicknesses of the green glass a good color match is not obtained when using a sector of small transmission. However, it is possible to repeat the setting quite well with two of these green glasses. For low temperatures it does not make much difference in the readings whether one or two glasses are used. This difference between the readings for two red glasses and for one red glass amounts to about one degree near 1800° K and to somewhat less at 1500° K, the reading being lower for one glass, for a tungsten pyrometer filament compared with a black body. For the blue glass for the same conditions the difference is somewhat greater, corresponding to about two degrees at 1800° K and to somewhat less at 1500° K. At lower temperatures this difference for either glass is much less. The accuracy gained by having the greater amount of light far outweighs the small error due to the change in effective wave-length.

Effective Wave-length of Monochromatic Screen.—An optical pyrometer can be calibrated and so used as to make unnecessary a knowledge of the extent to which the screen is monochromatic. To do this requires a black-body furnace that can be operated at various temperatures up to the highest tem-

perature for which the pyrometer is to be used. However, to use Wien's equation to extend the temperature scale, either above or below that of the standard furnace, by the use of a rotating sector disk or an absorbing glass, that is, to find the temperature of a black body having a brightness of, say, ten times (assuming a sector or absorbing glass transmission of one-tenth) that of a black body whose temperature can be measured directly, a knowledge of what wave-length to use, or the effective wave-

FIG. 2.



Effective wave-length for Jena red glass.
Curve *A*, effective wave-length from 1300°K to other temperatures.
Curve *B*, effective wave-length from 1800 to other temperatures.
Curve *D*, effective wave-length from 2400 to other temperatures.
Curve *E*, effective wave-length from 3600 to other temperatures.
Curve *C*, limiting effective wave-length.

length,* is necessary. The effective wave-length also must be known if the pyrometer is used to measure the temperature of non-black bodies. In using the pyrometer, it is the integral luminosities through the monochromatic screen that are compared, for which reason the effective wave-length of the screen corresponding to a certain temperature interval has been defined as the wave-length for that temperature interval of a black body, such that the ratio of its radiation intensities equals the ratio of the integral luminosities through the screeen used.

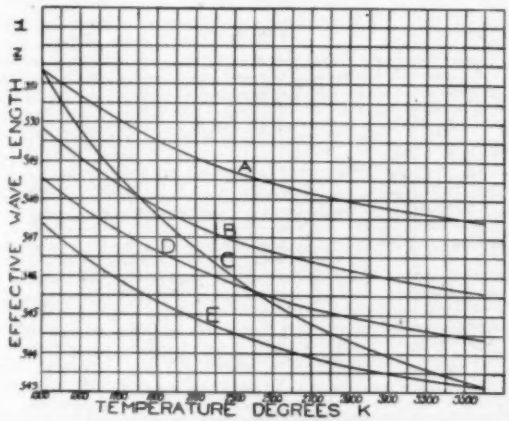
* *Astrophys. Jour.*, **42**, p. 294, 1915.

Knowing the spectral transmission of the glass, it is possible to calculate the effective wave-length λ_e for any temperature interval by means of the following equation:

$$\left[\frac{J(\lambda T_1)}{J(\lambda T_2)}\right]_{\lambda e} = \frac{\int_0^\infty J(\lambda T_1)\, V_\lambda\, t'\, d\lambda}{\int_0^\infty J(\lambda T_2)\, V_\lambda\, t'\, d\lambda}$$

where t' is the spectral transmission of the glass and V_λ is the visibility. These integrals also can be computed by the step-by-

FIG. 3.



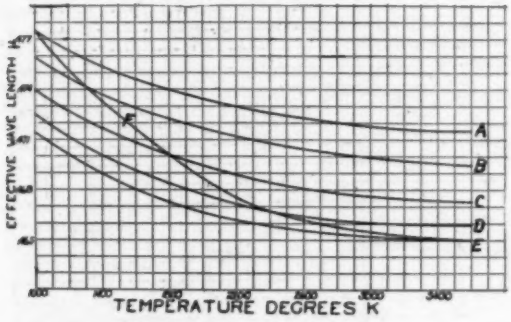
Effective wave-length of green glass.
Curve *A*, effective wave-length from 1300°K to other temperatures.
Curve *B*, effective wave-length from 1800 to other temperatures.
Curve *D*, effective wave-length from 2400 to other temperatures.
Curve *E*, effective wave-length from 3000 to other temperatures.
Curve *C*, limiting effective wave-length.

step method with sufficient accuracy for this purpose. Using this equation, the effective wave-length was calculated for each of the glasses whose spectral transmission is shown in Fig. 1 for a number of temperature intervals and the results plotted as shown in Figs. 2, 3 and 4. These curves in addition to those in Fig. 1 show the characteristics of the three glass screens. It will be noted in Fig. 2 that the effective wave-length for the red glass for the temperature interval 1800° K to 3000° K is 0.6659μ. The effective wave-length for the green and the blue glass screens for the same interval (Figs. 3 and 4) are respectively 0.5478μ and

0.4677μ. If it is desired to find the effective wave-length for some other temperature interval—as 2000° K to 2900° K—for which the curve is not drawn, the curve can be imagined drawn parallel to the 1800° K curve, one point being determined where the curve for the limiting effective wave-length crosses the ordinate for this temperature. The desired effective wave-length will be given by the point where this curve would cross the ordinate for the other temperature, in this case, the 2900° K ordinate.

When making a set of readings with the optical pyrometer the method used is first with the pyrometer filament darker than the background to get a brightness match by increasing the pyrometer current. This is called a dark reading. The next brightness match is made starting with the pyrometer filament brighter than the background. This setting is called a bright reading. The average of a dark and a bright is what is called a reading. Three or four such averages constitute a set. The readings that are used to study the accuracy consist of two or three such sets where each set consists of the average of six separate brightness matches. The different sets were obtained by making first a set with the red glass, then one with the green glass, and finally a set with the blue glass. These were then repeated in the same order. A study of the readings thus obtained showed that for brightnesses

FIG. 4.



Effective wave-length for blue glass.
Curve *A*, effective wave-length from 1000°K to other temperatures.
Curve *B*, effective wave-length from 1300 to other temperatures.
Curve *C*, effective wave-length from 1800 to other temperatures.
Curve *D*, effective wave-length from 2400 to other temperatures.
Curve *E*, effective wave-length from 3600 to other temperatures.
Curve *F*, limiting effective wave-length.

corresponding to temperatures above 1400° K or 1500° K the range in the averages of the currents for the condition of balance was about the same, whether a red, green or blue glass was used. The percentage range in the readings are given in Table II.

TABLE II.

Percentage Range in Current Through Pyrometer Filament for Brightness Matches with the Background Under Various Conditions and for Different Colored Screens in the Eyepiece.

	Screen		
	Red	Green	Blue
	per cent.	*per cent.*	*per cent.*
No. 1. Average range in pyrometer current for forty different determinations of temperature, 1200°K to 3000°K, each consisting of two or three different groups of six settings each.....	0.12	0.15	0.19
No. 2. Range for a single reading consisting of a dark and bright setting for forty determinations, temperature range 1200°K to 3000°K.....	.47	.42	.51
Same as No. 1, excepting for temperatures below 1500° K. 9 determinations.....	.12	.13	.36
Same as No. 2, excepting for the nine readings below 1500° K.....	.35	.28	.90

For the temperature range covered by this investigation, the relation between the percentage variation of the current through the pyrometer filament and the corresponding percentage variation in the temperature of the source being investigated for a brightness match is very nearly a constant independent of the temperature. It was also found that the constant was about the same for readings made with either of the three glasses. The result was that $\frac{dI}{I}=\frac{1.8\ dT}{T}$ where the constant 1.8 does not vary more than a few per cent. in going from the red to the blue. Thus, the per cent. variation in current divided by 1.8 will give the per cent. variation in the temperature. As can be seen from the data in Table II, the average range for the pyrometer currents for all temperature measured was very nearly the same for the readings with each of the three glasses and amounted to about 0.15 per cent. This makes the average range in the temperature measured less than one-tenth of a per cent., or less than 2° C for a temperature of 2000° K.

The data in Table II show that for temperatures below 1500° K, the readings with the blue glass had a range about three times as great as the readings with the red or green glass.

In the table is given also the average range due to a single measurement consisting of the average of a dark and a bright reading. The average range in pyrometer current for the forty temperature determinations was about 0.5 per cent., which makes the temperature range due to a single measurement about 5° at 2000°. Some of the readings, of course, showed a range somewhat greater than this. It must be remembered that this range corresponds to the greatest difference between any two readings (consisting of the average of a dark and a bright brightness match) for eight or ten readings. These results are thought to be pretty good and to indicate about the accuracy to be expected in this class of work. The results on the green glass show that the average for a single reading is less for the low temperatures than for the average of all the readings taken. This is due to the fact that many of the readings at high temperature were taken with a rotating sector and trouble due to poor color match was experienced. When a sector was used the current through the pyrometer was such that it corresponded to a temperature about 1500° K or higher.

To find out how different observers would agree in their readings, using the different colored screens, a number of readings were made by different observers. The brightness temperature of a tungsten lamp filament was measured at four different temperatures, using in turn each of the three glasses. The use of a tungsten lamp gives a greater brightness for the green and blue than would be obtained if a black body had been used at a temperature the same as this red brightness temperature. The brightness as observed at the blue end was increased by using a larger opening before the eyepiece telescope of the pyrometer, since it was found that readings could be made with this larger opening. The opening used with the red and green glasses was 9 mm in diameter, while for the blue glass the opening was 12 mm in diameter. Seven observers made readings for the test. The first four, as they are arranged in Table III, had had much experience with the optical pyrometer, using different colored screens. Observer No. 5 had had considerable experience with a commercial form of the disappearing-filament pyrometer using red glass. Observer No. 6 had had a small amount of experience, having read once before in a test of this kind. Observer No. 7 had had no previous experience with this kind of work. The first readings were made

on the lamp at a temperature of 1774° K, then at 2154° K, and finally at 1456° K. At a later date the apparatus was again set up and most of the observers made readings with the tungsten lamp at a brightness temperature of 1245° K. The temperature 2154° K was measured, using a 2° rotating sector disk between the pyrometer lamp and the lamp filament being measured. This made the current readings for the 2154° K point of the same order as for the 1456° K.

TABLE III.

Settings by Individual Observers Using Different Monochromatic Glass Screens.

Observer	Screen used	Current through pyrometer filament for apparent brightness match with tungsten filaments at the following brightness temperatures			
		1245°K direct	1456°K direct	1774°K direct	2154°K 2° sector
WEF	Red glass	2669	3353	4773	3197
AGW	Red glass	....	3358	4775	3193
CHS	Red glass	2665	3362	4780	3188
IAV	Red glass	2671	3357	4778	3190
FH	Red glass	2667	3356	4774	3212
FG	Red glass	2672	3363	4780	3185
HP	Red glass	2670	3351	4770	3199
	Mean	2669	3358	4776	3195
WEF	Green glass	2696	3393	4855	3619
AGW	Green glass	....	3394	4852	3609
CHS	Green glass	2687	3393	4863	3615
IAV	Green glass	2693	3398	4854	3617
FH	Green glass	2692	3398	4858	3631
FG	Green glass	2710	3384	4858	3605
HP	Green glass	2691	3410	4841	3615
	Mean	2695	3394	4854	3616
WEF	Blue glass	2720	3407	4882	3974
AGW	Blue glass	....	3413	4884	3969
CHS	Blue glass	2695	3414	4888	3982
IAV	Blue glass	2710	3412	4888	3966
FH	Blue glass	2707	3418	4894	3961
FG	Blue glass	2795	3429	4893	3979
HP	Blue glass	2722	3406	4766	3901
	Mean	2725	3414	4885	3971

These readings were made by the method described above. Each observer made but a single set of readings, consisting of six brightness matches at a particular temperature. The average of the current obtained by each observer is given in Table III. It will be noticed that the agreement is very good for the three highest temperatures and about the same for each of the screens used. For the lower temperatures the best agreement among the

different observers is found in the case where the red glass was used; also the agreement for the green glass is much better than for the blue glass.

For the readings on the 1240° K point a change of about 0.0003 amp. in the pyrometer current corresponds to one degree in temperature and for the 1456° K point a change of 0.0004 amp. in pyrometer current corresponds to about one degree in temperature. For the 1774° K point a change of one degree in temperature requires a change of 0.0005 amp. in pyrometer current, while for the 2154° K point a change of 0.0004 amp. corresponds to a change of about 3°. The reason for this greater change is due to the use of the rotating sector with small transmission.

From the results of the tests shown for all readings except at low temperatures it would be pretty hard to decide whether the green or red glass screen was the better, but the data show very definitely that the readings with the blue glass are not so good as those with the red or green. For very low temperatures it was almost impossible to make readings with the blue glass. For temperatures below 1200° K, and thus very low brightnesses, there is no question but that the red glass is the best to use. For temperatures in the neighborhood of 1400° K, the readings with the red and green glass have about the same range and this is somewhat greater than 0.1 of one per cent. The readings for the blue glass are very erratic and the range in some cases is several times this amount, the average being about three or four times this amount. Excepting at very low temperature (*i.e.*, below say 1200° K) there is no question of the amount of light obtained since either the red or green glass transmitted sufficient light for brightness matching. Personal preference will have considerable influence on this decision. The observers who made readings on these tests were about equally divided in their preference for the red or green glass screens. To the author, the fact that good color matches are not obtained under all conditions with the green glass would be against this glass.

There are four reasons for choosing the red glass rather than the green. In the first place, at low temperatures, the red radiation first becomes visible and thus readings may be made at a somewhat lower temperature with the red glass. This is well shown in Table I. In the second place, the color change for a change in wave-length is much less in the red part of the spectrum

than in the green, which makes the red the better part of the spectrum for this work. This is shown by the fact that for the green glass here used, a good color match is not obtained when a sector of small transmission is used, while no such trouble is found with the red glass. The third reason for using red glasses as monochromatic screens is that better red screens can be obtained, that is, the spectral transmission of the red screens ends with a steeper slope than does the green screen. This is illustrated very well by the relative slopes of the curves showing the spectral transmission of the red and green glass shown in Fig. 1. These curves show that the transmission of the red glass is very steep on the short wave-length side and the other side being at the limit of the visible spectrum, the very rapid falling off of the visibility curve in this region makes the luminosity curve steep on that side.

The fourth advantage of the red screen is that when measuring a temperature so high that a rotating sector or absorbing glass is necessary, the transmission of the sector or absorbing glass that must be used is larger for the longer wave-length than it is for the short wave-length. From Wien's equation the following relation can be derived between the wave-length and the transmission (t) of the sector used for extrapolating to a higher temperature.

$$\frac{\log t_1}{\log t_2} = \frac{\lambda_1}{\lambda_2}$$

Suppose that it is necessary to measure a temperature of 3100° K, using a pyrometer lamp or other comparison source that cannot be heated to a higher brightness than that corresponding to 1800° K. If the red glass described above is used, the transmission of the absorbing glass or sector that must be used is about 0.0054. For the green glass this same temperature range requires a sector or absorbing glass with a transmission of about 0.0017, while a sector or absorbing glass with a transmission of about 0.00061 is required for the blue glass described above. Such small transmissions are very hard to measure and thus it is better to use the red glass when measuring very high temperatures.

NELA RESEARCH LABORATORIES,
NELA PARK, CLEVELAND, OHIO,
December, 1920.

THE USE OF THE ULBRICHT SPHERE IN MEASURING REFLECTION AND TRANSMISSION FACTORS.*

BY

ENOCH KARRER.

CONTENTS.

	PAGE
I. INTRODUCTION	96
II. HISTORICAL	97
III. THE USE OF THE SPHERE	98
IV. THE THEORY AND USE OF INFINITE PLANES	103
V. THEORY OF THE SPHERE	108
VI. DESCRIPTION OF THE PRESENT REFLECTOMETER	111
VII. GENERAL COMMENTS ON METHODS OF MEASURING REFLECTION AND TRANSMISSION FACTORS, AND ON STANDARDS	114
VIII. A SIMPLER INSTRUMENT FOR COMMERCIAL USE	115
IX. A TRANSMISSOMETER	116
X. USE OF REFLECTOMETER AND TRANSMISSOMETER WITH MONOCHROMATIC LIGHT	119

I. INTRODUCTION.

THE purpose of this article is to present a general survey of the theory of the methods of measurement of the reflection factor of substances, and to describe in detail some new pieces of apparatus for the measurement of this quantity and for the measurement of the transmission factor.[1]

The apparatus described in greatest detail is a reflectometer consisting of a photometric sphere and the Martens polarization photometer. The novelty in this suggestion lies not in the use of this particular photometer in the study of the transmission and reflection by bodies, nor in the use of the sphere in this connection, but in the use of the sphere in a way that is in accord with the simple theory of the sphere, and that, in conjunction with a photometer such as the Martens, allows an absolute deter-

* Published by permission of the Director, Bureau of Standards.

[1] The reflection and the transmission factors for light are formally defined by the Illuminating Engineering Society (Rep. of Com. on Nomenclature and Standards, 1918). The terms are here used in accordance with these definitions, with the understanding, however, that they are determined under conditions of diffused light. Other symbols used in this article conform also with the I. E. S. nomenclature.

mination of either the transmission or reflection factor in one observation.

II. HISTORICAL.

The measurement of reflection factors for surfaces that reflect more or less diffusely has received considerable attention. Thaler [2] made a thorough test of the behavior of a few substances in this respect. Although he was not interested in the total diffuse reflection factors, his data [3] for a surface smoked with magnesium oxide determine values between .89 and .90 for such a surface.

Sumpner [4] made use of the increased brightness of the walls resulting when a lamp was introduced within an inclosure to study reflection and transmission by substances. Rosa and Taylor [5] used the same phenomenon with the sphere to obtain the reflection factor of their sphere paint. The method depends upon the comparison of illumination at a particular point from a given source when the direct flux only is considered and when this flux is altered by enclosing the source. Since the enclosure affects the flux in all directions from the source, the direct flux, with which comparison is made, must be the average direct flux from the source.

Gardner [6] gives data obtained in the Electrical Testing Laboratories by the use of the sphere in a manner suggested by Little, on the reflection factors of colored surfaces. This method is relative, consisting of the measurement of the brightness of the sphere walls when illuminated by the test or standard surface placed successively inside the sphere. The test or standard surface only is illuminated directly, by a source that is most conveniently placed outside the sphere.

Nutting [7] has made some interesting and important considerations on an instrument for determining the reflecting power of opaque bodies. This instrument was a very portable reflec-

[2] *Ann. d. Phys.*, **316**, p. 996; 1903. Good for reference to earlier work pertaining to Lambert's law. See also in this connection Mascart Traite D'Optique, Vol. 3, p. 213.

[3] *Loc. cit.*, p. 1006, Table I. (There appears to be a typographical error here in the record of angles.)

[4] "The Diffusion of Light," *Phil. Mag.*, **35**, p. 81; 1893.

[5] "The Integrating Photometric Sphere, Its Construction and Use," *Tr. I. E. S.*, **2**, p. 453; 1916.

[6] "The Light Reflecting Values of White and Colored Paints," *J. F. I.*, **181**, p. 99; 1916.

[7] *Tr. I. E. S.*, Vol. 7, p. 412; 1912.

tometer, depending upon no standards and had the unique property of requiring no light source as a necessary adjunct. Any general illumination at the place where measurements were made might suffice. Because this reflectometer bears some semblance to that described in this article, some parts of the underlying theory are given below. The sources of error that exist in the Nutting reflectometer will not be dwelt upon here. They have been pointed out by others.[9, 11]

III. THE USE OF THE SPHERE.

Luckiesh[8] has more recently called attention to the unsatisfactory state of the measurement of reflection factors, pointing out the desirability of arriving at some standard methods, and suggesting the use of the sphere in the following manner. Around a translucent sphere are distributed several light sources, and the whole surrounded by white-walled box. The surface whose reflection factor is sought fills a small aperture in the sphere. The brightness of the test surface is measured by matching the brightness of it with that of the field in an auxiliary photometer. The proposed method is good, but it involves the use of all the photometric accessories required in the ordinary use of the sphere, and requires further a standard reflecting surface. The means employed of illuminating the sphere walls should have to be altered somewhat in order that full advantage may be taken of the theory of the sphere. In any case, however, the use of the principle of the sphere has decided advantages over the laborious method of measuring the reflection factor for various directions, then by integration, graphically or otherwise, derive the total reflection factor.

Much credit is due Taylor[9] for pointing out several ways by which the sphere may be used to determine the absolute reflection factor. The most obvious ways (including those of Mr. Taylor and others) of using the sphere to measure the reflection factor may be enumerated as follows:

(1) Comparison of the illumination due to reflected flux alone with that due to both direct and reflected flux. If E_1 is the

[8] "Measurement of the Reflection Factor," *Elec. World*, Vol. 69, p. 958; 1917.

[9] "Measurement of Diffuse Reflection Factors," JOUR. AM. OPT. SOC., 4, p. 9; 1920. Also *B. S. Sci. Papers, No. 391* and *No. 405*.

illumination due to the average direct flux and E_2 that due to the reflected flux then

$$\frac{E_2}{E_1 + E_2} = \rho$$

This is the method used by Sumpner and Rosa and Taylor as described above to obtain the absorption factor (α) directly and derive therefrom the reflection factor. In this case

$$\frac{E_1}{E_1 + E_2} = \alpha = 1 - \rho$$

Sumpner, however, used an enclosure in the form of a hollow cylinder rather than in the form of a hollow sphere. Sumpner also used this method to obtain the transmission factor of substances. It is applicable only to materials that can be made into the form of a sphere, or that may be applied to the inner sphere walls.

(2) Sphere walls illuminated by illuminating a small portion of them. Brightness of the walls is measured when complete and when a known portion of the sphere is removed. This is the basis of a proposed reflectometer by Taylor [10] to obtain the reflection factor of painted sphere walls. The reflection factor of the sphere wall is given by the equation

$$\rho^2\left[q\,(p+q-1) + \frac{b}{b_o}p\,(1-q)\right] + \rho\,(1-q)\left[q + (p+q-1)\,(1-\frac{b}{b_o})\right] + (1-q)^2\,(1-\frac{b}{b_o}) = 0$$

where q is per cent of sphere surface cut off by the hole, p, the per cent of area added to the sphere wall when the hole is covered by a plane surface; b and b_o are brightnesses of sphere wall due to reflected light when test surface has a reflection factor of ρ and o respectively. A particular application of this method was made by Benford [11] for the determination of the reflection factor of magnesium carbonate as follows. The brightness (b_1 and b_2) of the sphere wall was measured after the removal in succession of two known fractional parts ($1 - p_1$) and ($1 - p_2$) of the sphere. The reflection factor is given by

$$\rho = \frac{b_1 - b_2}{b_1\,p_1 - b_2\,p_2}$$

[10] *Loc. cit.*, p. 16.

[11] *G. E. Rev.*, **23**, p. 72; 1920.

This method, as the previous one, is only applicable when the test substance can be made into a sphere or can be applied to the inner sphere walls. It has one very desirable feature, in that not only is the illumination diffuse but the reflected flux which is measured is also, and the whole of it is directly contributory to the results. The degree to which the flux is diffused depends upon the test substance itself, for the latter constitutes the whole of the sphere wall. It must be applied with caution when the test substances are any other than highly diffusing.

(3) The method (2) above may be modified so that the specimen need not be the whole of the sphere wall but merely a small known portion of it. In this case the reflection factor ρ_x of the test surface is given by

$$\rho_x = \frac{1 - \frac{b_o}{b_x}}{\frac{q}{1-q} \cdot \frac{b_o}{b_x} + \frac{\rho p}{\rho p + (1-q)(1-\rho)}}$$

where p, q, ρ, and b_o have the same meaning as in the method (2), and b_x is the brightness of sphere wall when the test surface is in place. The equation given under method (2) is derived from the present one where $\rho_x = \rho$. This method retains the desirable diffuseness in both incident and reflected light and obviates the restrictions of method (2) above, but the sensitivity of it depends upon the value of the reflection factor measured.

(4) Part Sphere.—Test (or standard) surface constituting a known portion of the sphere wall is illuminated directly. The standard is a specimen having the reflection factor of the sphere wall. The brightness of the sphere wall is measured first when illuminated by the specimen, then when illuminated by the standard. The reflection factor (ρ_x) of the test surface is given by [10]

$$\frac{\rho \frac{b_x}{b}}{1 - \frac{\rho^2 p \left(1 - \frac{b_x}{b}\right)}{1 - q + \rho (p + q - 1)}}$$

b_x and b are the values of the brightness of the sphere wall when the aperture is covered by the test surface and by the standard

surface respectively. This method was one adopted by Taylor in preference to the previous one, because it is more accurate and more sensitive.

(5) The sphere is complete. The specimen placed inside and illuminated directly (preferably by a narrow beam of light projected through a window). The standard is a specimen of known reflection factor. Two measurements of brightness of the sphere wall are necessary—first, when illuminated by the specimen, then when illuminated by the standard. This method was suggested and used by Little [6] whose standard was a magnesium carbonate surface. The ratio of these gives the reflection factor of the sample in terms of the standard. This method may be improved by eliminating the standard surface. This may be accomplished by screening the test surface from the observation window; then comparing the brightness of the sphere wall first, when illuminated by the sample, and second, when illuminated by illuminating a small portion of the wall. In the latter case no screen is interposed.

(6) The sphere is complete. The specimen placed inside and illuminated by the sphere wall. The brightness of the sample and of the sphere wall compared by one observation. The sphere wall may be illuminated by illuminating a small portion of it. The reflection factors are in terms of that of the sphere wall. If the test surface only is screened from the illuminated portion, the method becomes an absolute one.

(7) The specimen constitutes a negligible portion of the sphere wall, and is illuminated directly. The walls are illuminated by the specimen. The brightness of the sphere wall at a point screened from the test surface is measured first as resulting from the illumination of the test surface, second as resulting from an equal illumination of a portion of the sphere walls unscreened from the observed spot. The ratio of these is the reflection factor of the surface. This is the method embodied in the reflectometer more recently designed by Taylor.[12] No standard is required.

(8) Specimen constituting a negligible portion of the sphere is illuminated by sphere walls, which are illuminated by a narrow

[12] "A Simple Portable Reflectometer," A. H. Taylor, *Tr., I. E. S.* Paper read at Fourteenth Annual Convention of *I. E. S.*, 1920. Also *B. Sci. Papers, No. 405.*

beam admitted through a small window. The specimen is screened from the illuminated spot. No standard is required. This method was used by Sharp and Little.[26] These authors showed that the reflection factor of a mirror as well as that of other surfaces may be correctly measured by this method. It may also be pointed out that all movements of the photometer in making the measurements required (viz., of the brightness of first the specimen and second the sphere wall) may be avoided by employing a mirror of known reflection factor. In some experimental work with this method by the author a glass prism was used by the rotation of which either the specimen or the comparison spot could be brought into view.

(9) The method just described may be simplified by the use of a Martens Polarization Photometer. One measurement, comparing the brightness of the specimen with that of the sphere wall, will then suffice. This is the method to which special attention is called in this article. It is the method embodied in the reflectometer described in detail below. It is suggestive of the best combination (for certain photometric work) in which the sphere is used in a manner that takes advantage of its properties and at the same time affords simplicity of parts, compactness and directness.

Still other methods may be designed. For example:

(10) Specimen may constitute a known portion of sphere wall. Another known portion removed. Two measurements of brightness of sphere wall are required, first when sphere is complete, and second, when the known portion is removed. This has the advantage that both the incident and the reflected light are diffused. The degree to which the light is diffused depends in this case and in method (3) above not upon the specimen as in methods (1) and (2) above, but upon the sphere wall.

(11) Most of the above methods may be modified so that only one measurement is necessary by employing two spheres simultaneously.

(12) A new method of obtaining the reflection factor of a substance that may be applied to the walls of the sphere, or that may be made into a sphere is suggested by the equation given below under the theory of the sphere. The relation between the brightness (b_s) of the sphere wall resulting when a portion (p)

of it is made luminous and the brightness (b) of the luminous portion is given by

$$b_s = pb \cdot \frac{\rho}{1-\rho}$$

The reflection factor $\rho = \frac{r}{p+r}$, where $r = \frac{b_s}{b}$. The quantities p, b, and b_s may be measured. In particular, if p were changed until $b_s = b$, then $\rho = \frac{1}{1+p}$. It seems that this method is simpler than method (2) where portions of the sphere are removed, and simpler than method (1) where it is required to know the mean spherical candles of the source.

One simple way which suggests itself, whereby no beam of light need be projected into the sphere; viz., a plate of diffusing glass is illuminated on one side so that the brightness of the reverse side is b. The plate with the latter side toward the sphere is then placed against the sphere aperture of known dimensions. Finally the brightness of the sphere walls produced by this luminous plate is measured. This arrangement would readily allow of a variable aperture into the sphere.

Furthermore, the method of illuminating either the specimen or the sphere wall in some instances may be different from that stated above, for example, in method (8) above, the sphere wall might be illuminated by transmitted light as indicated by Luckiesh.[8]

IV. THE THEORY AND USE OF INFINITE PLANES.

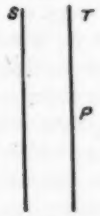
The Nutting instrument is based upon the principle of the infinite luminous plane. The illumination on a plane exposed to the flux [13] from an infinite parallel plane source of uniform brightness b (Lambert's) is πb, provided every element of the infinite plane radiates according to the cosine law. If a small reflecting surface be exposed to this illumination, it will reflect per unit area at the rate of $\pi\rho b$ lumens, where ρ is the reflection factor of the surface. Its brightness therefore is ρb. Hence the ratio of the brightness of the small plane to that of the infinite plane is the reflection factor sought. The surface under test should be small enough not to change the brightness of the luminous plane. Such simple conditions are not realized, however, in practice. In case of Nutting's reflectometer, two planes of the same size constitute

[13] *B. S. Bulletin*, Vol. 7, p. 553; 1910.

the radiating and the receiving planes. In such a case, if we assume them infinite, their resulting brightness is readily calculable.

The brightness of the planes is increased by the multiple reflection of light between them. Let S (Fig. 1) be the infinite plane of brightness b, illuminating the infinite plane T, and let the reflection factors of these planes be ρ_2 and ρ_1 respectively. As a result of the direct illumination from S, T will be of uniform brightness $\rho_1 b$, and will reradiate onto S at the rate $\pi\rho_1 b$. As a result of this S will have its brightness increased by an amount $\rho_1\rho_2 b$. The

FIG. 1.



Infinite planes.

plane S will in turn increase its illumination on T by an amount, $\pi\rho_1\rho_2 b$. The total illumination E_T of surface T will be

$$E_T = \pi b + \pi\rho_1\rho_2 b + \pi\rho_1^2\rho_2^2 b + \cdots + \text{etc.}$$

The brightness of T is therefore,

$$b_T = \rho_1 b\,(1 + \rho_1\rho_2 + \rho_1^2\rho_2^2 + \cdots + \rho_1^{n-1}\rho_2^{n-1} + \cdots +) = \rho_1 b/(1 - \rho_1\rho_2).$$

For the surface S we obtain the brightness

$$b_S = b\,(1 + \rho_1\rho_2 + \rho_1^2\rho_2^2 + \cdots + \rho_1^{n+1}\rho_2^{n+1} + \cdots +) = b/(1 - \rho_1\rho_2).$$

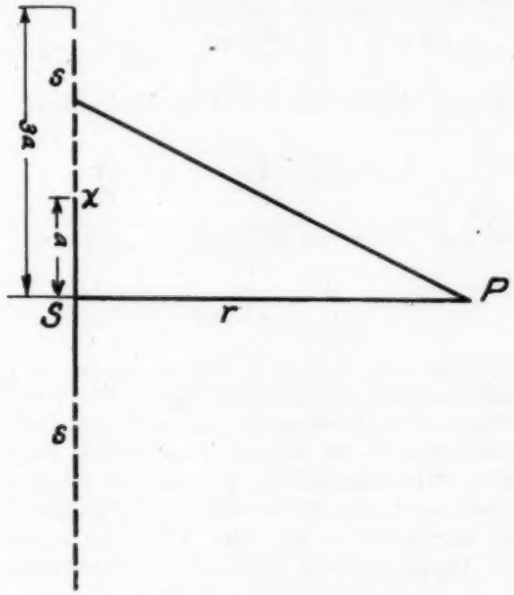
$$b_T/b_S = \rho_1.$$

In order to approach the condition of infinite planes, and at the same time retain limits consistent with a portable instrument, Nutting bounded the planes with a mirror.

The effect of the mirror is virtually to extend by multiple reflection the planes indefinitely. If the mirror were non-absorbing the finite planes bounded by it would give effects similar to

infinite planes. The theory of them then would be also as simple as that of infinite planes. The mirror, however, will reflect only a certain proportion ρ_m of the flux falling upon it, and as a consequence, instead of an infinite plane with uniform brightness, we have an infinite plane composed of concentric zones of varying brightness. It becomes more difficult to calculate the illumination

FIG. 2.



A disk S surrounded by annular areas ss of lesser brightness.

due to such a luminous plane at any point in space. This will be seen by considering, for sake of simplicity, the illumination at a point P which lies on the line perpendicular to the plane at the common centre of all the zones. Let S, Fig. 2, be a luminous disk of diameter 2a and of uniform brightness b, surrounded by an annular area of uniform brightness ρ_mb. For the present the annular area may be regarded as the first reflection of the disk in a surrounding circular mirror which reflects an amount ρ_m.

The width of the annular area is therefore 2a; its brightness $\rho_m b$. The illumination due to the disk S at the point P is

$$E_d = \pi b \int_0^a \frac{2r^2 x\,dx}{(r^2+x^2)^2} = \frac{\pi b a^2}{r^2+a^2}$$

where r is the distance of the point P from the disk.[14] Similarly the illumination at P due to the annular portion *ss* of the disk of brightness b_1 is

$$E_1 = \pi b_1 \int_a^{3a} \frac{2r^2 x\,dx}{(r^2+x^2)^2} = \frac{8\pi b_1 a^2 r^2}{(r^2+9a^2)(r^2+a^2)}.$$

So that the total illumination at P is the sum of these

$$E_t = \frac{\pi b a^2}{(r^2+a^2)}\left(1 + \frac{8\rho_m r^2}{r^2+9a^2}\right).$$

where $\rho_m = b_1/b$.

If the disks were of uniform brightness, *i.e.*, $\rho_m = 1.00$,

$$E_t = 2\pi b r^2 \int_0^{3a} \frac{x\,dx}{(r^2+x^2)^2} = \frac{9\pi b a^2}{(r^2+9a^2)}.$$

In the actual case there is an infinite number of concentric zones which contribute to the illumination at P and the resultant effect at P may be represented by a summation of such integrals as that given above for the annular zone *ss* due to the first order of reflection. The effect of the n^{th} order of reflection may be written after noting that the position and boundary of the n^{th} image are determined by its radii, which are $a(2n-1)$ and $a(2n+1)$. The illumination at P due to the n^{th} reflection is then

$$E_n = \pi b \rho_m^n \int_{a(2n-1)}^{a(2n+1)} \frac{2r^2 x\,dx}{(r^2+x^2)^2} = \frac{8\pi b a^2 r^2 n\,\rho_m^n}{\left[r^2+(2n-1)^2a^2\right]\left[r^2+(2n+1)^2a^2\right]}$$

The illumination due to all zones is

$$E_a = \sum_{n=1}^{n=\infty} \pi b \rho_m^n \int_{a(2n-1)}^{a(2n+1)} \frac{2r^2 x\,dx}{(r^2+x^2)^2} = \sum_{n=1}^{n=\infty} \frac{8\pi b a^2 r^2 n\,\rho_m^n}{\left[r^2+(2n-1)^2a^2\right]\left[r^2+(2n+1)^2a^2\right]}.$$

[14] *B. S. Bulletin*, Vol. 6, p. 551; 1910.

And the total illumination becomes

$$E_t = E_d + E_a = \pi b a^2 \left\{ \frac{1}{r^2 + a^2} + 8r^2 \sum_{n=1}^{n=\infty} \frac{n\rho_m^n}{[r^2+(2n-1)^2a^2][r^2+(2n+1)^2a^2]} \right\}.$$

A few terms of this expression may be written out for the sake of the discussion.

$$E_t = \pi b a^2 \left[\frac{1}{a^2+r^2} + 8r^2 \left\{ \frac{\rho_m}{(r^2+9a^2)(r^2+a^2)} + \frac{2\rho_m^2}{(r^2+25a^2)(r^2+9a^2)} + - - + \right\} \right.$$

$$+ 8r^2 \left\{ \frac{n\rho_m^n}{[r^2+(2n-1)^2a^2][r^2+(2n+1)^2a^2]} \right.$$

$$\left. \left. + \frac{(n+1)\rho_m^{n+1}}{[r^2+(2n+3)^2a^2][r^2+(2n+1)^2a^2]} + - - + \right\} \right].$$

This series is in general very rapidly converging. The ratio of the $(n+1)^{th}$ to the n^{th} term being,

$$\rho_m \cdot \frac{n+1}{n} \left\{ 1 - \frac{(8+16n)a^2}{r^2+(2n+3)^2a^2} \right\},$$

which approaches the value ρ_m for large values of n, but is always less than ρ_m if a is large compared with r (say $a = 3r$ or more). The value of this ratio for the first and second terms is ρ_m if $a = 2r$. The effect of the illumination due to the n^{th} reflection compared with that of the disk alone is

$$e = \frac{8r^2 n \rho_m^n (a^2+r^2)}{[r^2+(2n+1)^2a^2][r^2+(2n-1)^2a^2]}.$$

If a/r is assumed to be 5, and ρ_m to be 100, then the effect of the first and second images is about 5 per cent. of that of the disk and the effect of the first 10 images is slightly less than 6 per cent. of that of the disk. It is evident that as the reflection factor of the mirror approaches unity the disk as extended by its images becomes more like an infinite plane.

From the equation for E_t it is also seen that as r becomes very small the illumination at P approaches that given by an infinite plane of uniform brightness, $E_t = \pi b$.

The foregoing considerations show that the theoretically simple conditions are not at all attained in the use of infinite planes as in the Nutting instrument. An improvement could be effected by making the test surface only a small portion of one of the planes. This would also be the case if one were to consider the instrument from the point of view of a finite enclosure rather than from the viewpoint of infinite planes.

A very good approach to two infinite planes can be had by two concentric spherical surfaces. Either one of the surfaces may be made luminous with uniformity of brightness. The test surface may be made a part or the whole of the non-luminous sphere. A sphere uniformly bright illuminates all points on any given concentric spherical surface equally, and more nearly like an infinite plane as the points approach the surface of the luminous sphere.[15] Of course the last condition would be impossible to realize in a practical instrument. Yet, this interposes no hindrance to the practical use of two concentric spherical surfaces as described. For when the surfaces are separated any finite distance the relative distribution of the illumination from the one on to the other remains unaltered. The magnitude only of the illumination changes, becoming less than that which is given by an infinite plane in the ratio of the square of the radius of the sphere to the square of the distance from the centre of the sphere to the concentric spherical surface. (The latter statement applies when the inner sphere is the luminous one.) Just what other difficulties would arise in the actual execution of this arrangement cannot be foretold. It is of interest to note that the flux between two infinite parallel planes depends upon the reflection factors of them in a manner quite different from that in which the flux in a sphere depends upon the reflection factor of the sphere wall.

V. THE THEORY OF THE SPHERE.

For an enclosure we find the most satisfactory case to consider is that of the hollow sphere, for although the flux in any enclosure is everywhere the same,[16] if the walls of the enclosure are uniformly bright, it is only in case of the sphere where this latter condition of uniformity of brightness is easily obtainable.

In the sphere, whose walls reflect according to Lambert's

[15] *B. S. Bulletin*, Vol. 6, p. 558; 1910.

[16] *B. S. Bulletin*, Vol. 6, No. 4, p. 562.

law, every element of surface illuminates every other element to the same degree,[17] no matter how the wall of the sphere or any portion of it is directly illuminated. This property of the spherical enclosure was first discovered by Sumpner.[4] It was rediscovered by Ulbricht,[18] who first demonstrated its great practical importance to photometry. So that, if the surface of the sphere has a uniform reflection factor and, if a small portion of it is made luminous, the whole sphere will be uniformly bright, except for this spot which may be of negligible proportions. The brightness may be calculated as follows:

Let the illuminated spot of area A have an average brightness of b. It radiates πbA lumens. Owing to multiple reflections this flux will accumulate to a value F until the rate of loss αF equals the rate of input πbA. Where α is the absorption coefficient,

$$\alpha F = \pi bA.$$

$$F = \frac{\pi bA}{\alpha} = \frac{\pi bA}{(1-\rho)}$$

This was first derived by Mascart.[19] This reasoning may be employed for any enclosure whose walls obey Lambert's cosine law of emission. When the walls are not uniformly bright, b refers to the average brightness.

The illumination on every other element of the surface of the sphere will be

$$E = \frac{bA}{4r^2} \cdot \frac{1}{1-\rho}.$$

Hence the brightness of every element of the surface of the sphere except A is

$$\frac{\rho bA}{4\pi r^2} \cdot \frac{1}{1-\rho} = b_s.$$

The ratio of the area of the luminous spot to the area of the sphere is

$$\frac{A}{4\pi r^2} = p$$

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[17] Liebenthal, *Praktische Photometrie*, p. 301.

[18] *Electroteknische Zeit.*, 21, p. 595; 1900, and 26, p. 512; 1905.

[19] Palaz, "Traite de Photométrie Industrièlle," Trans. by the Pattersons, p. 297.

So that

$$b_s = \frac{pb\,\rho}{1-\rho}.$$

If now for a small portion of the sphere surface there is substituted another non-luminous surface whose reflection factor is ρ_x; the brightness b_x of the latter will be

$$b_x = \frac{\rho_x\, bp}{1-\rho} = \frac{\rho_x\, b_s}{\rho}.$$

$$b_x / b_s = \rho_x / \rho.$$

The reflection factor of any specimen is given in terms of that of the sphere wall. To make it possible to obtain the reflection factor of the specimen absolutely, certain conditions must be imposed upon the manner in which the specimen is illuminated. If all of the flux illuminating the specimen is reduced by an amount ρ, the brightness of the specimen will also be reduced in the same proportion.

Then b_x becomes $b_x' = \rho\, b_x = \rho_x\, b_s$.
and

$$\frac{b'_x}{b_s} = \rho_x.$$

This result is achieved by interposing a screen between the illuminated spot and the test surface. The screen is just sufficiently large to enable it to intercept all of the flux reflected from the former in the direction of the latter. The illuminated spot must be small in order that no appreciable illumination due to the second, and higher order of reflections would come to the test surface, if the screen were not present. The new surface also is assumed to be so small that its presence does not alter the brightness of the sphere walls.

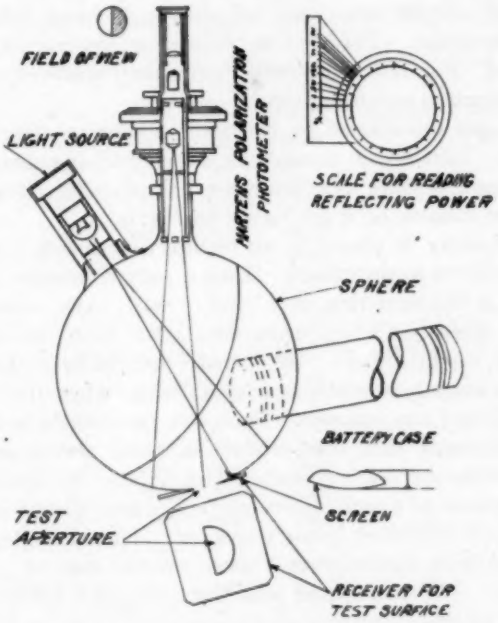
All of these theoretical conditions can be attained to a satisfactory degree in practice. This is the idea underlying several uses of the sphere in reflectometry listed above and in particular the reflectometer [20] depicted in Fig. 3.

[20] This reflectometer was shown before the Illuminating Engineering Society at Cleveland, Oct. 5, 1920. A general description of it will appear in *Trans. I. E. S.*, as discussion of the papers: "Measurement of Reflection Factor," C. H. Sharp and W. F. Little; and "A Simple Portable Reflectometer of the Absolute Type," A. H. Taylor.

VI. DESCRIPTION OF THE PRESENT REFLECTOMETER.

The reflectometer sketched in Fig. 3 is essentially a sphere, having, firstly, an aperture which may be closed by any surface whose reflecting factor is sought; secondly, a means for comparing directly the brightness of the test surface with that of the

FIG. 3.



The sphere reflectometer using the Koenig-Martens polarization photometer.

sphere surface such as a Martens [21] photometer; and thirdly, a lamp for illuminating a small portion of the sphere.

The most satisfactory photometer for comparing the brightness of two surfaces not far removed from each other is the Martens Polarization Photometer. It is very compact, accurate, and has an extremely good photometric field.[22] Its optical parts

[21] *Physikalische Zeitschrift*, Vol. 1, pp. 299–303; 1900.

[22] It is to be regretted that this instrument is not more readily procurable.

are in order of position in the optical path, Fig. 3, two apertures, lens, Wollaston prism, biprism, Nicol prism, eyepiece containing lens and aperture. A scale reading directly the absolute reflection factor is attached to the circular scale already a part of the Martens Photometer. The lamp is a small mazda flashlight lamp operated at 3.8 V. by a small 3-cell flashlight battery. It is not difficult to get sufficient flux into the sphere and at the same time illuminate a rather small spot, say 3 cm in diameter, when proper lenses are chosen. For the first instrument constructed no lenses were used. For ease and accuracy of setting, however, the resulting illumination was insufficient.

The spot illuminated by the beam is screened from the test aperture. This screen is made no larger than is necessary. Its dimensions and shape were determined graphically. The size and shape and location of it are shown in Fig. 3.

The battery is placed in an ordinary flashlamp tube whose end is soldered to the sphere. Thus a very convenient handle is afforded at the same time.

The sphere is a copper-covered steel float, procurable at plumbers' supply houses. Its diameter is 6 inches. The copper coat was actually found to be undesirable, when the sphere is enamelled, and was removed. The screen between the test aperture and illuminated spot was riveted in place and enamelled as the rest of the surface. A coat of white enamel facilitates greatly the attainment of a satisfactory surface, where a matt white surface of high reflection factor is desired. The enamelled surface is easy to paint and requires a much thinner coat of paint than otherwise. The paint used was that described by Mr. A. H. Taylor [23] of this Bureau.

A few tests are recorded here to indicate the behavior of an instrument constructed along these lines. These tests were made with the first instrument, constructed rather hastily and imperfectly. No lenses were used with the lamp; no adequate holder for the battery attached; and no provision made for readily turning the Martens Photometer through 180° to eliminate any asymmetric errors. Settings were made in each of the four quadrants.

[23] *B. S. Sci. Papers*, in press, "The Integrating Sphere," Rosa and Taylor.

Magnesium Carbonate Block No. 1 (A. H. T.).

	Quadrant				
	First	Second	Third	Fourth	Reflection Factor.
	43.9	134.5	223.2	313.6	
	43.3	134.2	223.9	313.9	
	43.9	133.9	223.3	314.1	
	43.7	134.4	223.1	314.2	
	43.8	133.8	223.2	314.3	
Mean	43.7	134.1	223.3	314.0	.982

Magnesium Carbonate Block No. 2 (A. H. T.).

	Quadrant				
	First	Second	Third	Fourth	Reflection Factor.
	43.1	133.8	223.6	314.8	
	43.2	133.9	223.2	314.7	
	44.0	134.1	223.2	314.3	
	43.5	134.2	223.1	314.5	
	43.2	134.3	223.3	313.7	
				314.2	
Mean	43.4	134.1	223.3	314.3	.972

Magnesium Carbonate Block No. 1(E. K.).

	Quadrant				
	First	Second	Third	Fourth	Reflection Factor.
	44.0	134.1	223.0	313.5	
	44.1	134.6	223.5	314.1	
	43.8	133.9	223.2	314.0	
	43.3	134.5	223.5	313.9	
	43.2	134.0	223.2	313.7	
	43.5		223.2		
Mean	43.6	134.2	223.3	313.8	.987

The first two blocks of magnesium carbonate were loaned by Mr. Taylor, who has determined the reflection factor of them by various methods, arriving at .99 for No. 1 and .982 for No 2. There seems to be a consistent difference, yet it is small and almost within experimental errors.

Other tests were made on a series of surfaces which had been studied by Mr. Taylor using the point-by-point method. Differences in the factor were usually obtained (although not always), which indicated that the above reflectometer was reading slightly low. The value obtained here for the magnesium carbonate

agrees with those found first by Mr. Taylor [24] and later by Mr. Benford [25] and others.[26]

The sample needs to be neither opaque nor diffusing. It may be a clear piece of glass or mirror. This condition follows on the assumption that the hole in the sphere over which the specimen is placed is so small that its area is negligible in comparison with that of the whole sphere. In the one described above, the area of the hole is about 3 square cm, or less than one-half per cent. of that of the sphere. In the case of a mirror, however, precautions must be taken not to have the second reflecting surface too far removed from the opening. The angle subtended by the image of the test aperture must always be sufficiently large to fill the field of view of the photometer.

VII. GENERAL COMMENTS ON METHODS OF MEASURING REFLECTION AND TRANSMISSION FACTORS, AND ON STANDARDS.

Many aspects of the problem of measuring the diffuse reflection factor call for consideration, some of which may or may not turn out to be of any moment. For instance, in case of the reflection factor, the surface may be diffusely illuminated, but in general can be observed from one direction only. Or it may be illuminated from one direction only, and the diffusely reflected flux measured. In either case the question of direction enters, concerning which nothing much can be said until more work is done; the use of the sphere is, however, a step toward uniformity and standardization. In case of the transmission factor the illumination on the specimen may be entirely diffuse and the whole of the transmitted flux measured. This can be accomplished by the use of two spheres, and comparing brightness of the walls of them. So that the question of direction may be easily eliminated in case of the transmission factor.

Four of the methods enumerated above for determining the reflection factor, namely (2), (3), (9), and (12), afford diffusion

[24] A. H. Taylor: "Measurement of Diffuse Reflection Factors and a New Absolute Reflectometer," JOUR. AM. OP. SOC., Vol. 4, No. 1, p. 9; 1920.

[25] F. A. Benford: "An Absolute Method for Determining Coefficients of Diffuse Reflection," *Gen. Elec. Rev.*, Vol. 23, p. 72; 1920.

[26] C. H. Sharp and W. F. Little: "Measurement of Reflection Factors," *Trans. I. E. S.* Paper read at Fourteenth Annual Convention, *I. E. S.*, 1920. See also paper by Mr. A. H. Taylor, presented at the same time; and discussion by Karrer.

in both the incident and the measured flux. The first and last of these places very serious limitations upon the nature and the form of the surface to be tested. The other two are insensitive for certain values of the reflection factor, their sensitivity depending upon this factor in a rather complicated manner. Method (1) virtually allows diffuseness in both incident flux and measured flux, but the extent to which this condition actually prevails depends upon the relative importance of the secondary flux (E_2), and therefore upon the reflection factor itself.

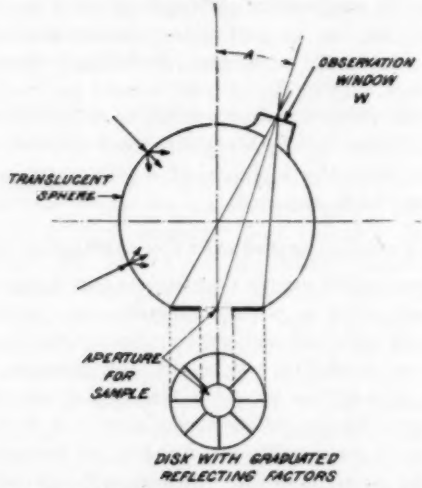
At all events, so far as a working standard is concerned, the procuring of a magnesium carbonate block at any market is to be discouraged so far as preliminary observations go. Magnesium carbonate blocks purchased at different times may have reflection factors varying by at least several per cent. It would be an admirable thing for some scientific supply house to have in stock magnesium blocks of guaranteed texture and purity. Yet those who have the facilities of a photometric sphere need not rely on even such standards.

VIII. A SIMPLER INSTRUMENT FOR COMMERCIAL USE.

The Nutting reflectometer requires neither lamp nor battery, a great advantage in a portable instrument. This advantage could possibly be obtained with other reflectometers by sacrificing accuracy for serviceability. It might, for example, be possible to accomplish this by the use of a translucent sphere or one in part transparent. In the reflectometer shown in Fig. 3 a mirror mounted so as to have sufficient freedom of rotation was substituted for the lamp to obviate the necessity of maintaining a light source. Unless, however, an intense external light source is available, the brightness is too low. With sunshine it works admirably. For practical uses, however, even simpler instruments might be used; for example, the one shown in Fig. 4. In this an opal glass sphere has a rather large opening covered with a flat disk. The disk has a circular aperture against which the test surface is placed, surrounded by an annular area divided into sectors, having different known reflection factors. The test surface and sectors are equally illuminated by the sphere wall, and are viewed through an aperture, W. The brightness of the test surface is taken to be that of the sector whose brightness most nearly matches it. The use of a simple disk with graduated

reflection factors was suggested many years ago. It is not satisfactory where any great amount of specular reflection exists. In the present instrument, such is not the case. Furthermore, a definite direction determined by angle A, Fig. 4, for observing is given. This direction might be chosen most favorably for many common materials met with in practice. Several workers [27] seem to find that at some angle of emergence for normal illumination many surfaces have a reflection factor nearly equal to that of a perfectly diffusing surface of the same diffuse reflection factor.

FIG. 4.



A simple and inexpensive reflectometer for commercial use.

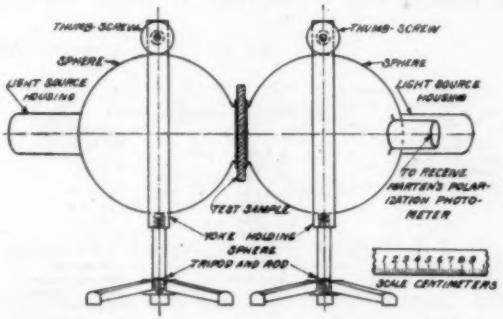
IX. TRANSMISSOMETER.

The diffuse transmission factor is sometimes more important than the reflection factor, as of photographic negatives and in diffusing glassware for illumination purposes. It is therefore desirable to have at hand a means of readily measuring this quantity with a fair degree of accuracy. While there are several ways in which the properties of the Ulbricht Sphere may be taken advantage of for this purpose, let it suffice to point out in detail

[27] A. H. Taylor finds this angle to be about 50° for many surfaces. Prof. W. J. Drisko informs me that he has found the angle to be about 46° for many surfaces.

here only one such way which is merely an extension of that involved in the reflectometer described above. Another similar sphere with an aperture equal to that in the reflectometer sphere, as is shown in Fig. 5, is added. The second sphere is illuminated by the same lamp or by one similar to, and in series with, the one illuminating the first sphere. For a stationary piece of apparatus where no restrictions are placed on size, the same lamp could illuminate both spheres. The flux into the second sphere is controlled by means of a diaphragm, with a variable aperture, such as an iris, slit, or wedge (see Fig. 6). No exacting specifications are necessary. The aperture of the second sphere is screened from the direct flux, as in the reflectometer sphere. In this case, as in

FIG. 5.



An absolute transmissometer using two spheres.

the reflectometer, we assume that the area of the aperture is so small that it will not effect the brightness of the sphere walls. This can probably always be sufficiently well attained. It is also assumed that the illuminated area is small.

The use of this apparatus as a transmissometer is as follows: The transmission sample is placed over the aperture of the reflectometer as usual and its reflection factor ρ measured. The auxiliary sphere is then brought up so that the reverse side of the specimen covers its aperture. The brightness b_x of the sample now is due to two factors: (1) the flux reflected from its front surface (which is illuminated by the walls of the first sphere); and (2) the flux transmitted by the specimen in consequence of the illumination by the walls of the second sphere. Let the walls

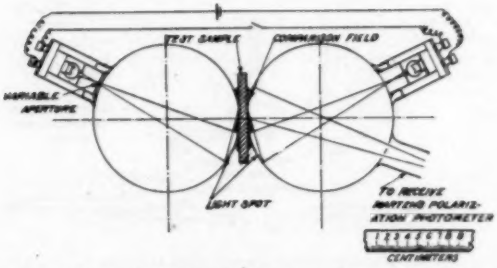
of these spheres have a brightness of b and b_1, respectively, and let the transmission factor of the sample be τ. Then $b_x = \rho b + \tau b_1$. The quantity b_1 may be measured in terms of b once for all, and may have any value, but for the sake of simplicity, we shall make $b_1 = b$.

Then $b_x = b(\rho + \tau)$.

$$b_x/b = \rho + \tau.$$

In this equation ρ has been determined; b_x is measured in terms of the brightness of the walls of the reflectometer sphere; τ is the factor desired; and $\frac{b_x}{b}$ is measured and read off directly by means of the reflectometer. We see therefore that it is just as simple so far as manipulation is concerned to obtain the value of the

FIG. 6.



The transmissometer in horizontal section; showing the position of the light source, test sample, photometer and screens.

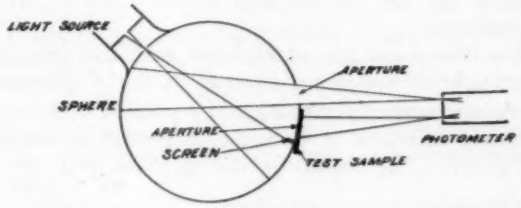
combined reflection and transmission factors as it is to obtain the reflection factor alone. The difference of these two quantities gives the transmission factor. In this way one piece of laboratory apparatus enables us to obtain both the reflection factor and the transmission factor in an absolute manner, involving no calibration of a standard of transmission or reflection, and no light standards, no measurement of distances, no manipulation of the specimen, no electrical measuring instruments. One placing of the sample against the reflectometer is all.

Although the arrangement described appears to have certain advantages for laboratory purposes where duplication of costly photometric or spectrophotometric apparatus is not desired, or

where it is desirable that such apparatus when once set up shall not be disturbed, it may be of interest to describe a few other arrangements that may be had. A simple method would be to insert the sample between the end of the photometer and sphere in such a way that one of the apertures only of the photometer is covered by it. The other aperture of course is illuminated by the sphere wall, which in turn is illuminated by projecting a narrow beam into the sphere. Precautions must be taken to screen the test sample properly as has been noted in the description of the reflectometer. Narrow limits are set to the size and shape of the sample that is to be tested.

A modification of this is shown in Fig. 7, where the photometer is removed from the sphere to a distance that allows

FIG. 7.



An absolute transmissometer using one sphere only and the Koenig-Martens polarization photometer.

greater freedom in the size and shape of the sample. In this case a comparison is made of the brightness b of the sphere wall with the brightness τb of the sample. The Martens polarization photometer measures τ directly.

In all these arrangements nothing novel can be claimed except in the use of the sphere rather than of an irregular inclosure or of superficial standards. The sphere affords two important essentials: (1) the kind of diffuse illumination desired in such measurements, and (2) the conditions for absolute determination. Neither of these can be so closely approximated by any other means.

The combination of the sphere with the Martens Photometer has been suggested and used by Goldberg [28] in measuring the density of photographic plates. His suggestions differ from those

[28] Photographische Korrespondenz, May–June, 1910. Densograph, ein Registrierapparat zur Messung der Schwarrzung photographischer Platten.

given in this article regarding the general use of the sphere in measuring the transmission and reflection factors of substances in that in accordance with the latter the sphere is to be used not only to secure a good diffuseness of the light but to obtain these factors in an absolute manner.

X. USE OF REFLECTOMETER AND TRANSMISSOMETER WITH MONOCHROMATIC LIGHT.

Although the instruments discussed above were primarily intended for mixed light, obviously they may also be used in determining the diffuse reflection and transmission factors of substances for spectral light. For this purpose the Koenig-Martens spectrophotometer is admirably adapted. In this connection, too, the use of the sphere, in ways described above, would make for definiteness and standardization, and afford absolute methods.

Most of this work was carried out with the coöperation of the Searchlight Investigation Section of the Engineers, U. S. Army, who also made the executing of it possible by providing for it. Gratitude therefore is sincerely expressed to them. I am indebted to Mr. U. M. Smith for the illustrations of the text.

BUREAU OF STANDARDS,
WASHINGTON, D. C.

NOTICES.

As the result of the recent election, the following members of the Optical Society of America have been elected officers for the year 1921:

President, J. P. C. SOUTHALL.
Vice-President, C. E. MENDENHALL.
Secretary, IRWIN G. PRIEST.
Treasurer, ADOLPH LOMB.
Members at large of the Council:
P. G. NUTTING,
C. E. K. MEES,
L. A. JONES,
W. E. FORSYTHE.

(Signed) F. L. MOHLER, *Teller.*
PAUL D. FOOTE, *Secretary.*

The Council has appointed F. K. Richtmyer as the official representative of the Optical Society in the American Association for the Advancement of Science.

Members of the Optical Society should notify the Secretary of change in address. Examine the address on the envelope containing the JOURNAL, and if incorrect please advise.

The membership of the Optical Society is now more than twice as great as that of one year ago and the subscription list to the JOURNAL has increased by three times. In order to continue this rate of growth it is necessary for each member to secure at least one additional member during the next few months. Please make special effort to secure new members so that the success of the society and of its JOURNAL may be insured.

RECENTLY ELECTED MEMBERS OF THE OPTICAL SOCIETY.

(Continued from Page 504, Vol. IV, No. 6, Nov., 1920.)

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Conrad August Stone, 521 Washington Ave., Charleroi, Pa.
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Herman W. Farwell, Physics Dept., Columbia University, New York, N. Y.

*Associate membership.

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